

ACTIONS OF TREES ON SEMIGROUPS, AND AN INFINITARY GOWERS–HALES–JEWETT RAMSEY THEOREM

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ABSTRACT. We introduce the notion of (Ramsey) action of a tree on a (filtered) semigroup. We then prove in this setting a general result providing a common generalization of the infinitary Gowers Ramsey theorem for multiple tetris operations, the infinitary Hales–Jewett theorems (for both located and nonlocated words), and the Farah–Hindman–McLeod Ramsey theorem for layered actions on partial semigroups. We also establish a polynomial version of our main result, recovering the polynomial Milliken–Taylor theorem of Bergelson–Hindman–Williams as a particular case. We present applications of our Ramsey-theoretic results to the structure of delta sets in amenable groups.

1. INTRODUCTION

The finitary *Hales–Jewett theorem* [14] is a fundamental combinatorial pigeonhole principle. Several years after the original proof of Hales and Jewett, two deep infinitary strengthenings of the Hales–Jewett theorem (for located and nonlocated words) have been proved in [4] using the theory of ultrafilters and algebra in the Stone–Čech compactification. In another direction, and around the same time, Gowers established in [12] another fundamental combinatorial pigeonhole principle, which has been since then referred to the (infinite) *Gowers Ramsey theorem*. Gowers’ Ramsey theorem is a far-reaching generalization of *Hindman’s theorem on finite unions* [15]. We refer to [13, 17, 20, 24, 25] for other proofs of such a result and its finitary counterpart, where explicit bounds on the quantities involved are also obtained.

A common generalization of Gowers’ Ramsey theorem and the infinite Hales–Jewett theorems has been established by Farah–Hindman–McLeod in the setting of *layered actions* on *adequate partial semigroups* [10]. In a different direction, the infinite Gowers Ramsey theorem has been strengthened in [18] by considering *multiple tetris operations*. This answered a question of Bartošová and Kwiatkowska from [2], where the corresponding finitary statement is proved. A common generalization of Gowers’ Ramsey theorem for multiple tetris operations and the *Milliken–Taylor theorem* [19, 23] is also provided in [18].

Gowers’ Ramsey theorem for multiple tetris operation does not fit in the framework of layered actions on partial semigroups developed by Farah–Hindman–McLeod in [10]. It is therefore natural to wonder whether there exists a unifying combinatorial principle that lies at the heart of both Gowers’ Ramsey theorem for multiple tetris operations and the infinite Hales–Jewett theorems, as well as the Farah–Hindman–McLeod Ramsey theorem for layered actions on partial semigroups. The goal of the present paper is to provide such a unifying combinatorial principle within the framework, here introduced, of *Ramsey actions of rooted trees on filtered semigroups*. Our main result is Theorem 5.10, which provides a common generalization of all the results mentioned above. One can also obtain from such a general result more direct common generalizations of Gowers’ theorem for multiple tetris operations and the Hales–Jewett theorems (for located and nonlocated words). Such common generalizations—Theorem 4.5 and Theorem 5.12—are stated in terms of variable words with *variables indexed by a finite rooted tree*, and variable substitution maps that respect the tree structure. We also provide a common generalization of our main result—Theorem 5.10—and the *polynomial Milliken–Taylor theorem* of Bergelson–Hindman–Williams [7, Corollary 3.5]; see Theorem 6.2. All the results of this paper are infinitary, and imply by a routine compactness argument their finitary counterparts. We omit the statement of these finitary counterparts, leaving it to the interested reader. We will conclude by presenting applications of some of our Ramsey-theoretic results to the structure of delta sets in amenable graphs.

The present paper consists of six sections, besides this introduction. In Section 2 we introduce and study the notion of action of an ordered set and of a rooted tree on a compact right topological semigroup. Section 3 deals with the notion of (Ramsey) action of an ordered set and of a rooted tree on a partial semigroup. General result

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for Ramsey actions of rooted trees on adequate partial semigroups is obtained here (Theorem 3.2 and Corollary 3.7). Section 4 explains how Gowers' theorem for multiple tetrises operations and the Hales–Jewett theorem are both subsumed by Theorem 3.2. Section 5 considers the even more general framework of (Ramsey) actions of rooted trees on filtered semigroups. It is explained here how all the previous results extend to this more general framework. This allows one to recover the infinite Hales–Jewett theorem for nonlocated words. Section 6 presents a further polynomial generalization, which subsumes this main result of the paper as well as the polynomial Milliken–Taylor theorem [7, Corollary 3.5]. Finally, Section 7 presents applications to combinatorial configurations contained in delta sets in amenable groups.

After the present paper was written, we have been informed that a general Ramsey statement subsuming Gowers' theorem for multiple tetrises operations and the infinitary Hales–Jewett theorems has been independently obtained by Solecki with different methods. We refer the reader to [22] for this alternative approach.

In the rest of this paper we denote by ω be the set of natural numbers including 0, and \mathbb{N} be the set of natural numbers different from zero. We identify an element n of ω with the set $\{0, 1, \dots, n-1\}$ of its predecessors. If A, B are finite nonempty subsets of ω , we write $A < B$ if the maximum element of A is smaller than the minimum element of B . We also write $A < \ell$ for $A \subset \omega$ and $\ell \in \omega$ if the largest element of A is smaller than ℓ . Given a set A we let $[A]^{<\aleph_0}$ be the set of finite subsets of A . If D is a set, then we denote by βD the space of ultrafilters on D ; see [16, Chapter 3]. This is endowed with a canonical compact Hausdorff topology, having the sets $\overline{A} = \{\mathcal{U} \in \beta D : A \in \mathcal{U}\}$ for $A \subset D$ as basis of open (and closed) sets. We will use in the rest of the paper the notation of *ultrafilter quantifiers*; see [24, Chapter 1]. If $\psi(x)$ is a formula depending on a variable x ranging over D , then we write $(\mathcal{U}x) \varphi(x)$ as an abbreviation for $\{x \in D : \varphi(x) \text{ holds}\} \in \mathcal{U}$. In particular, we have that $(\mathcal{U}x) x \in A$ is equivalent to the assertion that $A \in \mathcal{U}$. By a *finite coloring* of the set D we mean a function $c : D \rightarrow n$ for some $n \in \omega$. Any such a coloring admits a *canonical extension*, which we still denote by c , to a finite coloring of βD , obtained by setting $c(\mathcal{U}) = i$ if and only if $(\mathcal{U}x), c(x) = i$.

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2. ACTIONS OF TREES ON COMPACT RIGHT TOPOLOGICAL SEMIGROUPS

2.1. Compact right topological semigroups. We recall here some notions concerning compact right topological semigroups. An (additively denoted) compact right topological semigroup X is a semigroup $(X, +)$ endowed with a compact topology with the property that, for every $z \in X$, the right translation map $x \mapsto x + z$ is continuous. *In the following we assume all the compact right topological semigroup to be Hausdorff.* An element e of X is *idempotent* if $e + e = e$. A classical result of Ellis [9, Corollary 2.10]—see also [24, Lemma 2.1]—asserts that any compact right topological semigroup contains idempotent elements. One can define an order among idempotents of X by setting $e_0 \leq e_1$ if and only if $e_0 + e_1 = e_1 + e_0 = e_0$. An idempotent element of X is *minimal* if it is minimal with respect to such an order. The proof of [9, Corollary 2.10] also shows that for any idempotent element e_0 of X there exists a minimal idempotent e_0 of X such that $e \leq e_0$.

A closed subsemigroup A of X is a *nonempty* closed subset of X with the property that $x + y \in A$ whenever $x, y \in A$. Observe that the idempotent elements of X are precisely the closed subsemigroups of X that contain a single element. A closed subsemigroup A of X is a closed *bilateral ideal* if $x + a$ and $a + x$ belong to A whenever $a \in A$ and $x \in X$. We denote by $\mathcal{S}(X)$ the set of closed subsemigroups of X . We define an order in $\mathcal{S}(X)$ by setting $A \leq B$ if and only if $(A + B) \cup (B + A) \subset A$. Clearly a subsemigroup A of X is a bilateral ideal if and only if $A \leq X$. Observe that such an order extend the order on idempotents defined above, when an idempotent element e of X is identified with the closed subsemigroup $\{e\}$. If X is a compact right topological semigroup, we define $\text{End}(X)$ to be the set of continuous semigroup homomorphisms $\tau : X \rightarrow X$. Observe that $\text{End}(X)$ is a semigroup with respect to composition.

In the following we will regard $\mathcal{S}(X)$ as an ordered set endowed with such an ordering. (Here and in the following, all the ordered sets are supposed to be *partially* ordered.) We record here for future reference the following well known fact; see also [24, Lemma 2.3].

Lemma 2.1. Suppose that X is a compact right topological semigroup. If $A, B \in \mathcal{S}(X)$ and $A \leq B$, then for any idempotent $b \in B$ there exists a minimal idempotent a of A such that $a \leq b$.

Proof. Consider a minimal idempotent element a of $A + b$. Observe that $b + a$ is an idempotent element of $b + a$ such that $(b + a) + a = b + a$. Therefore by minimality of a inside $A + b$ we have that $b + a = a = a + b$ and hence $a \leq b$. Suppose now that z is an idempotent element of A such that $z + a = z$. Then we have that $z + b = z + a + b = z + a = z$ and hence $z \in A + b$. It follows from minimality of a inside $A + b$ that $z = a$. Therefore a is a minimal idempotent element of A . \square

2.2. Actions of trees on compact right topological semigroups. Suppose that \mathbb{P} is an ordered set, and X is a compact right topological semigroup.

Definition 2.2. An *action* α of \mathbb{P} on X is given by

- an order-preserving function $\mathbb{P} \rightarrow \mathcal{S}(X)$, $t \mapsto X_t$,
- a subsemigroup $\mathcal{F}_\alpha \subset \text{End}(X)$,

such that for every $\tau \in \mathcal{F}_\alpha$ there exists a function $f_\tau : \mathbb{P} \rightarrow \mathbb{P}$ —which we call the *spine* of τ —such that τ maps X_t to $X_{f_\tau(t)}$ for every $t \in \mathbb{P}$, and such that $\tau(x) = x$ for any $x \in X_t$ and $t \in \mathbb{P}$ such that $f_\tau(t) = t$.

Given an action α of \mathbb{P} on X we let X_α be set of functions $\xi : \mathbb{P} \rightarrow X$ such that $\xi(t) \in X_t$ and $\tau \circ \xi = \xi \circ f_\tau$ for every $\tau \in \mathcal{F}_\alpha$ and $t \in \mathbb{P}$. When X_α is nonempty, we endow X_α with the product topology and the entrywise operation. This turns X_α into a compact right topological semigroup. Observe that an idempotent in X_α is an element ξ of X_α such that $\xi(t)$ is an idempotent element of X_t for every $t \in \mathbb{P}$. We say that an idempotent ξ in X_α is *order-preserving* if $\xi(t_0) \leq \xi(t_1)$ whenever $t_0 \leq t_1$.

Suppose now that T is a rooted tree. We regard T as an ordered set endowed with the canonical rooted tree order obtained by setting $t' \leq t$ if and only if t' is a descendent of t .

Definition 2.3. A *regressive homomorphism* of T is a function $f : T \rightarrow T$ such that $f(t) \geq t$ for every $t \in T$, and f maps two adjacent nodes either to the same node or to adjacent nodes.

It is clear that any regressive homomorphism fixes the root, and maps every branch to itself.

Definition 2.4. A *Ramsey action* α of T on X is given by an action of T on X in the sense of Definition 2.2 such that X_α is nonempty and, for every $\tau \in \mathcal{F}_\alpha$, the corresponding spine $f_\tau : T \rightarrow T$ is a regressive homomorphism.

A similar proof as [18, Lemma 2.1] shows the following.

Proposition 2.5. Suppose that T is a rooted tree of height $\leq \omega$ with root r . If α is a Ramsey action of T on X , then X_α contains an order-preserving idempotent. Furthermore, if $\xi^{(0)}$ is an idempotent element of X_α , then X_α contains an order-preserving idempotent ξ such that $\xi(r) = \xi^{(0)}(r)$.

Proof. Fix an idempotent element $\xi^{(0)}$ of X_α . Let, for $k \in \omega$, $\pi_k : T \rightarrow T$ be the function that maps every node to its k -th predecessor, where we convene that the k -th predecessor of a node of height at most k is the root, and the 0-th predecessor of every node is itself. Let T_k be the set of nodes in T of height at most k . We define by recursion on $k \in \omega$ idempotent elements $\xi^{(k)}$ of X_α such that $\xi^{(k)}(t) + \xi^{(k)}(t_0) = \xi^{(k)}(t)$ whenever $t_0 \in T_{k-1}$, and $t \in T$ are such that $t \leq t_0$, and $\xi^{(k)}(t) = \xi^{(j)}(t)$ whenever $j \leq k$ and $t \in T_j$. Granted the construction one can then consider $\xi \in X_\alpha$ defined by

$$\xi(t) := (\xi^{(n)} \circ \pi_n + \xi^{(n)} \circ \pi_{n-1} + \cdots + \xi^{(n)} \circ \pi_0)(t)$$

for any node $t \in T$ of height n . It is not difficult to verify that $\xi \in X_\alpha$ is an order-preserving idempotent such that $\xi(t) = \xi^{(0)}(r)$.

We proceed now with the recursive construction. We have already defined $\xi^{(0)}$. Suppose that $\xi^{(0)}, \dots, \xi^{(k)}$ have been defined for some $k \in \omega$. Consider the closed subsemigroup Z_k of $\xi \in X_\alpha$ such that $\xi(t_0) = \xi^{(k)}(t_0)$ for $t \in T_k$, and $\xi(t) + \xi(t_0) = \xi(t)$ for every $t \in T$ and $t_0 \in T_k$ such that $t \leq t_0$. Observe that Z_k is nonempty. Indeed, set

$$\xi(t) := (\xi^{(k)} \circ \pi_0 + \xi^{(k)} \circ \pi_1 + \cdots + \xi^{(k)} \circ \pi_n)(t)$$

for any node $t \in T$ of height n . Observe that $\tau \circ \xi = \xi \circ f_\tau$ for every $\tau \in \mathcal{F}_\alpha$ since τ is a homomorphism, f_τ is an regressive homomorphism of T , and $\tau \circ \xi^{(k)} = \xi^{(k)} \circ f_\tau$ by recursive assumption. Therefore $\xi \in X_\alpha$. Furthermore if $t_0 \in T_k$, then $\xi^{(k)}(t) + \xi^{(k)}(\pi^{(j)}(t)) = \xi^{(k)}(t)$ for every $j \in \omega$ and hence $\xi(t) = \xi^{(k)}(t)$. Finally suppose that $t_0 \in T_k$ and $t \in T$ are such that $t \leq t_0$. We want to prove that $\xi(t) + \xi(t_0) = \xi(t)$. Suppose that the height of t is m . If $m \leq k$ then we have that

$$\xi(t) + \xi(t_0) = \xi^{(k)}(t) + \xi^{(k)}(t_0) = \xi^{(k)}(t) = \xi(t)$$

by the recursive assumption. Suppose now that $m > k$. Then we have that, by the recursive assumption,

$$\begin{aligned} \xi(t) + \xi(t_0) &= \xi^{(k)}(t) + \cdots + \xi^{(k)}(\pi_{m-k}(t)) + \xi^{(k)}(t_0) \\ &= \xi^{(k)}(t) + \cdots + \xi^{(k)}(\pi_{m-k}(t)) \\ &= \xi(t). \end{aligned}$$

This concludes the proof that $\xi \in Z_k$. Since Z_k is nonempty, it contains an idempotent element $\xi^{(k+1)} \in Z_k$. This concludes the recursive construction. \square

The following definition is inspired by the definition of layered action from [10, Definition 3.3].

Definition 2.6. An action α of T on X is a *layered action* if for every $\tau \in \mathcal{F}_\alpha$ and $t \in T$ one has that

- (1) f_τ is equal to either t or the immediate predecessor of t ;
- (2) if t has an immediate predecessor t^- , then for any minimal idempotent $p \in X_{t^-}$ there exists $q \in X_t$ such that $\sigma(q) = p$ for any $\sigma \in \mathcal{F}_\alpha$ such that $f_\sigma(t) = t^-$.

A similar proof as [10, Theorem 3.8] gives the following.

Proposition 2.7. Suppose that T is a rooted tree of height $\leq \omega$, and α is a layered action of T . Then α is a Ramsey action. Furthermore, X_α contains an order-preserving idempotent ξ such that $\xi(t)$ is a minimal idempotent in X_t for every $t \in T$.

Proof. It is clear by definition of layered action that, for every $\tau \in \mathcal{F}_\alpha$, f_τ is a regressive homomorphism. We now prove the second assertion. This will also show that α is a Ramsey action.

We define minimal idempotents $x_t \in X_t$ by recursion on the height of t such that, for every $t, t' \in T$ and $\tau \in \mathcal{F}_\alpha$, $\tau(x_t) = x_{f_\tau(t)}$ and $x_{t'} \leq x_t$ if $t' \leq t$. If t_0 is the root of T then we let x_{t_0} be any minimal idempotent element of X_{t_0} . Suppose that x_t has been defined whenever the height of t is at most k . Suppose now that t has height $k+1$ and let t^- be the immediate predecessor of t .

If, for each $\tau \in \mathcal{F}_\alpha$, $f_\tau(t) = t$, one can just define $x_t \in X_t$ using Lemma 2.1. Suppose now that $f_\tau(t) = t^-$ for some $\tau \in \mathcal{F}_\alpha$. Let Y be the set of $z \in X_t$ such that $\tau(z) = x_{t^-}$ for every $\tau \in \mathcal{F}_\alpha$ such that $f_\tau(t) = t^-$. By hypothesis we have that Y is nonempty. Observe now that $Y + x_{t^-} \subset Y$. Indeed we have that for $y \in Y$, $\tau(y + x_{t^-}) = x_{t^-} + x_{f_\tau(t)} = x_{t^-}$ by recursive hypothesis. Pick now a minimal idempotent x_t of $Y + x_{t^-}$. Observe that $x_{t^-} + x_t \in Y + x_{t^-}$ is an idempotent such that $x_{t^-} + x_t \leq x_t$. By minimality, $x_{t^-} + x_t = x_t = x_t + x_{t^-}$ and hence $x_t \leq x_{t^-}$. Observe that if $z \in Y$ is an idempotent element such that $z \leq x_t$ then $z \in Y + x_{t^-}$. This shows that x_t is minimal in Y . Finally suppose that $z \in X_t$ is an idempotent element such that $z \leq x_t$. Then we have that, for any $\tau \in \mathcal{F}_\alpha$, $\tau(z)$ is an idempotent element of X_{t^-} such that $\tau(z) \leq x_{t^-}$. It follows by minimality of x_{t^-} that $\tau(z) = x_{t^-}$, and hence $z \in Y$. Minimality of x_t in Y now shows that $z = x_t$. This concludes the proof that x_t is minimal in X_t . \square

3. ACTIONS OF TREES ON PARTIAL SEMIGROUPS

3.1. Partial semigroups. A *partial semigroup* [10, Definition 1.2]—see also [4, Section 2] and [24, Section 2.2]—is a set S endowed with a partially defined binary operation $(x, y) \mapsto x + y$, with the property that $(x + y) + z = x + (y + z)$ for $x, y, z \in S$. Such an equality should be interpreted as asserting that the left hand side is defined if and only if the right hand side is defined, and in such a case they are equal. A partial semigroup is *adequate* [10, Definition 2.1]—or *directed* [24, §2.2]—if for every finite subset A of S the set $\varphi_S(A)$ of elements x of S such that $a + x$ is defined for every $a \in A$ is nonempty.

A *partial semigroup homomorphism* [10, Definition 2.8] between partial semigroups S and T is a function $f : S \rightarrow T$ with the property that $f(x + y) = f(x) + f(y)$ for $x, y \in S$. Again, such an equality should be interpreted as asserting that the left hand side is defined if and only if the right hand side is defined, and in such a case they are equal. A partial semigroup homomorphism is *adequate* if for every finite subset B of T there exists a finite subset A of S such that the image of $\varphi_S(A)$ under f is contained in $\varphi_T(B)$. If $S \subset T$ then we say that S is an *adequate partial subsemigroup* of T if the inclusion map is an adequate partial semigroup homomorphism [10, Definition 2.10]. We say furthermore that S is an *adequate bilateral ideal* if it is an adequate partial subsemigroup, and, for every $x \in S$ and $y \in T$, $x + y, y + x$ belong to S whenever they are defined.

If S_0, S_1 adequate partial subsemigroups of S , then we let $S_0 \leq S_1$ if $(S_0 + S_1) \cup (S_1 + S_0) \subset S_0$. This should be interpreted as the assertion that, for any $s_0 \in S_0$ and $s_1 \in S_1$, $s_0 + s_1$ and $s_1 + s_0$ belong to S_0 whenever they are defined. Observe that $S_0 \leq S$ if and only if S_0 is an adequate bilateral ideal of S . We denote by $\mathcal{S}(S)$ the space of adequate partial subsemigroups of S . We regard $\mathcal{S}(S)$ as an ordered set with respect to the ordering just defined.

3.2. Cofinal ultrafilters on partial semigroups. Suppose that S is a partial semigroup. An ultrafilter \mathcal{U} over S is *cofinal* if $\forall x \in S, \mathcal{U}x, x + y$ is defined. Following [24, Chapter 2], we denote by γS the space of cofinal ultrafilters over S . It is clear that γS is a closed subspace of the space of ultrafilters over S . Furthermore, γS is a compact right topological semigroup when endowed with the operation defined by setting $A \in \mathcal{U} + \mathcal{V}$ if and only if $\mathcal{U}x, \mathcal{V}y, x + y \in A$; see [24, Corollary 2.7] and [10, Theorem 2.6]. More generally, this expression defines a function $\beta S \times \gamma S \rightarrow \beta S, (\mathcal{U}, \mathcal{V}) \mapsto \mathcal{U} + \mathcal{V}$ such that, for any $\mathcal{V} \in \gamma S$, the function $\beta S \rightarrow \beta S, \mathcal{U} \mapsto \mathcal{U} + \mathcal{V}$ is continuous. In particular, for any $s \in S$ and $\mathcal{U} \in \gamma S$, the element $s + \mathcal{U}$ of βS is well defined.

Suppose that S_0 and S_1 are partial semigroups, and $\sigma : S_0 \rightarrow S_1$ is an adequate partial semigroup homomorphism. Then σ induces a continuous semigroup homomorphism $\sigma : \gamma S_0 \rightarrow \gamma S_1$ by setting $A \in \sigma(\mathcal{U})$ if and only if $\mathcal{U}x$, $\sigma(x) \in A$ [10, Lemma 2.14]. When S_0 is an adequate subsemigroup of S_1 and $\sigma : S_0 \rightarrow S_1$ is the inclusion map, the continuous extension $\sigma : \gamma S_0 \rightarrow \gamma S_1$ is one-to-one. In this situation, we will identify γS_0 with its image under σ , which is the closed subsemigroup of γS_1 consisting of the cofinite ultrafilters on S_1 that contain S_0 . This defines a map $\mathcal{S}(S) \rightarrow \mathcal{S}(\gamma S)$, $S_0 \mapsto \gamma S_0$. Here $\mathcal{S}(\gamma S)$ denotes as in Subsection 2.1 the space of closed subsemigroups of γS . It is not hard to see that such a map is order-preserving with respect to the ordering on $\mathcal{S}(S)$ and $\mathcal{S}(\gamma S)$ defined above.

3.3. Actions of ordered sets on partial semigroups. Suppose that \mathbb{P} is an ordered set, and S is an adequate partial semigroup. We denote by $\text{End}(S)$ the space of *adequate* partial semigroup homomorphisms $\tau : S \rightarrow S$. Observe that $\text{End}(S)$ is a semigroup with respect to composition.

Definition 3.1. An *action* α of \mathbb{P} on S is given by

- an order-preserving function $\mathbb{P} \rightarrow \mathcal{S}(S)$, $t \mapsto S_t$, and
- a subsemigroup $\mathcal{F}_\alpha \subset \text{End}(S)$,

such that for every $\tau \in \mathcal{F}_\alpha$ there exists a function $f_\tau : \mathbb{P} \rightarrow \mathbb{P}$ —which we call the *spine* of τ —such that τ maps S_t to $S_{f_\tau(t)}$ for every $t \in \mathbb{P}$, and such that $\tau(s) = s$ for any $s \in S_t$ and $t \in T$ such that $f_\tau(t) = t$.

Suppose that α is an action of \mathbb{P} on S . Then α induces an action in the sense of Definition 2.2 of \mathbb{P} on the compact right topological semigroup $X = \gamma S$, which we still denote by α . This is obtained by setting $X_t := \gamma S_t$ for $t \in T$ and considering the semigroup of continuous semigroup homomorphisms $\tau : \gamma S \rightarrow \gamma S$ obtained as the canonical continuous extensions of elements τ of \mathcal{F}_α . Consistently with the notation introduced in Subsection 2.2, we denote by $(\gamma S)_\alpha$ the set of functions $\xi : \mathbb{P} \rightarrow \gamma S$ such that $\xi(t) \in \gamma S_t$ and $\tau \circ \xi = \xi \circ f_\tau$ for every $t \in \mathbb{P}$. An order-preserving idempotent in $(\gamma S)_\alpha$ is an element ξ of $(\gamma S)_\alpha$ such that $\xi(t)$ is an idempotent element in γS_t and $\xi(t) \leq \xi(t_0)$ whenever $t, t_0 \in \mathbb{P}$ are $t \leq t_0$.

Theorem 3.2. Suppose that α is an action of a finite ordered set \mathbb{P} on the adequate partial semigroup S . Suppose that $\xi \in (\gamma S)_\alpha$ is an order-preserving idempotent. Fix a finite coloring c of S and consider its canonical extension to a finite coloring c of βS . Fix a sequence $(\psi_n^{(\mathcal{F})})$ of functions $\psi_n^{(\mathcal{F})} : (S^\mathbb{P})^n \rightarrow [\mathcal{F}_\alpha]^{<\aleph_0}$ and a sequence $(\psi_n^{(S)})$ of functions $\psi_n^{(S)} : (S^\mathbb{P})^n \rightarrow [S]^{<\aleph_0}$ such that $\psi_n(x_0, \dots, x_{n-1})$ contains the range of $\tau_i \circ x_i$ for every $i \in n$ and $\tau_i \in \psi_i^{(\mathcal{F})}(x_0, \dots, x_{i-1})$. There exists a sequence (x_n) of functions $x_n : \mathbb{P} \rightarrow S$ such that

- $x_n(t) \in S_t \cap (\varphi_S \circ \psi_n^{(S)})(x_0, \dots, x_{n-1})$ for every $n \in \omega$ and $t \in \mathbb{P}$; and
- for any $\ell \in \omega$, $n_0 < n_1 < \dots < n_\ell \in \omega$, $t_i \in \mathbb{P}$ for $i \leq \ell$, and $\tau_i \in \psi_{n_i}^{(\mathcal{F})}(x_0, \dots, x_{n_i-1})$ for $i \leq \ell$, if $\{f_{\tau_i}(t_i) : i \leq \ell\}$ is a chain in \mathbb{P} with least element t , then the color of $\tau_0(x_{n_0}(t_0)) + \dots + \tau_\ell(x_{n_\ell}(t_\ell))$ is equal to the color of $\xi(t)$.

Proof. We now define by recursion on $m \in \omega$ functions $x_m : \mathbb{P} \rightarrow S$ such that $x_m(t) \in S_t \cap (\varphi_S \circ \psi_m^{(S)})(x_0, \dots, x_{m-1})$ such that for every $m \in \omega$ the following holds:

- (1_m) for every $\ell \leq m$, $n_0 < n_1 < \dots < n_\ell \leq m$, $t_i \in \mathbb{P}$ for $i \leq \ell$, $\tau_i \in \psi_{n_i}^{(\mathcal{F})}(x_0, \dots, x_{n_i-1})$ for $i \leq \ell$ such that $\{f_{\tau_i}(t_i) : i \leq \ell\}$ is a chain in \mathbb{P} with least element t , one has that the color of $\tau_0(x_{n_0}(t_0)) + \dots + \tau_\ell(x_{n_\ell}(t_\ell))$ is equal to the color of $\xi(t)$, and
- (2_m) for every $\ell \leq m$, $n_0 < n_1 < \dots < n_\ell \leq m$, $t_i \in \mathbb{P}$ for $i \leq \ell+1$, $\tau_i \in \psi_{n_i}^{(\mathcal{F})}(x_0, \dots, x_{n_i-1})$ for $i \leq \ell$ such that $\{f_{\tau_i}(t_i) : i \leq \ell\} \cup \{t_{\ell+1}\}$ is a chain in \mathbb{P} one has that the color of $\tau_0(x_{n_0}(t_0)) + \dots + \tau_\ell(x_{n_\ell}(t_\ell)) + \xi(t_{\ell+1})$ is equal to the color of $\xi(t)$.

Let us consider initially the case $m = 0$. In this case $(S^\mathbb{P})^0$ is a single point. Therefore $\psi_0^{(S)}$ selects a finite subset S_0 of S , and $\psi_0^{(\mathcal{F})}$ selects a finite subset \mathcal{F}_0 of \mathcal{F}_α . We need to find a function $x_0 : \mathbb{P} \rightarrow S$ such that $x_0(t) \in S_t \cap \varphi_S(S_0)$ for every $t \in \mathbb{P}$ and such that the following holds:

- (1₀) for every $t_0 \in \mathbb{P}$ and $\tau_0 \in \mathcal{F}_0$ the color of $\tau_0(x_0(t_0))$ is equal to the color of $\xi(f_{\tau_0}(t_0))$, and
- (2₀) for every $t_0, t_1 \in \mathbb{P}$ and $\tau \in \mathcal{F}_0$ if $\{f_\tau(t_0), t_1\}$ is a chain in \mathbb{P} with least element t , then the color of $\tau_0(x_0(t_0)) + \xi(t_1)$ is equal to the color of $\xi(t)$.

Fix $t \in \mathbb{P}$. Using the notation of ultrafilter quantifiers for the ultrafilter $\xi(t)$, we have that $\xi(t)s$, $\forall \tau_0 \in \mathcal{F}_0$, $\forall t_1 \in \mathbb{P}$ such that $\{f_{\tau_0}(t), t_1\}$ is a chain in \mathbb{P} with least element t_{\min} , one has that $s \in S_t \cap \varphi_S(S_0)$, the color of $\tau_0(s)$ is equal to the color of $\xi(f_{\tau_0}(t))$ and the color of $\tau_0(s) + \xi_1(t_1)$ is equal to the color of $\xi(t_{\min})$. This allows one to choose $x_0(t) \in S_t \cap \varphi_S(S_0)$ satisfying (1₀) and (2₀).

We now consider the case $m = 1$. In this case $\psi_0^{(\mathcal{F})}$ selects a finite subset \mathcal{F}_1 of \mathcal{F}_α , and $\psi_1^{(S)}$ selects a finite subset $S_1 = \psi_1^{(S)}(x_0)$ that contains $\tau_0(x_0(t))$ for every $\tau_0 \in \mathcal{F}_0$ and $t \in \mathbb{P}$. From (2₀) we deduce that

- (3₀) for every $t_0, t_1, t_2 \in \mathbb{P}$ and $\tau_0 \in \mathcal{F}_0$, if $\{f_{\tau_0}(t_0), t_1, t_2\}$ is a chain in \mathbb{P} with least element t , then the color of $\tau_0(x_0(t_0)) + \xi(t_1) + \xi(t_2)$ is equal to the color of $\xi(t)$.

Now fix $t \in \mathbb{P}$. We have that $\xi(t)s$, $\forall \tau_0 \in \mathcal{F}_0$, $\forall \tau_1 \in \mathcal{F}_1$, $\forall t_0, t_2 \in \mathbb{P}$, one has that $s \in S_t \cap \varphi_S(S_1)$, the color of $\tau_1(s)$ is equal to the color of $\xi(f_\tau(t))$, if $\{f_{\tau_0}(t_0), f_{\tau_1}(t)\}$ is a chain in \mathbb{P} with least element t_{\min} then the color of $\tau_0(x(t_0)) + \tau_1(s)$ is equal to the color of $\xi(t_{\min})$, if $\{f_{\tau_0}(t_0), f_{\tau_1}(t), t_2\}$ is a chain in \mathbb{P} with least element t_{\min} then the color of $\tau_0(x(t_0)) + \tau_1(s) + \xi(t_2)$ is equal to the color of $\xi(t_{\min})$, and if $\{\tau_1(t), t_2\}$ is a chain in \mathbb{P} with least element t_{\min} then the color of $\tau_1(s) + \xi(t_2)$ is equal to the color of $\xi(t_{\min})$. This allows one to choose $x_1(t) \in S_t \cap \varphi_S(S_1)$ in such a way that (1₁) and (2₁) are satisfied.

Suppose that a sequence as above has been defined up to m in such a way that (1_m) and (2_m) are satisfied. From (2_m) and the fact that ξ is an order-preserving idempotent in $(\gamma S)_\alpha$, it follows that the following holds as well:

- (3_m) for every $\ell \leq m$, $n_0 < n_1 < \dots < n_\ell \leq m$, $t_i \in \mathbb{P}$ for $i \leq \ell + 2$, $\tau_i \in \psi_{n_i}^{(\mathcal{F})}(x_0, \dots, x_{n_{i-1}})$ for $i \leq \ell$ such that $\{f_{\tau_i}(t_i) : i \leq \ell\} \cup \{t_{i+1}, t_{i+2}\}$ is a chain in \mathbb{P} with least element t , one has that the color of $\tau_0(x_{n_0}(t_0)) + \dots + \tau_\ell(x_{n_\ell}(t_\ell)) + \xi(t_{\ell+1}) + \xi(t_{\ell+2})$ is equal to the color of $\xi(t)$.

Fix $t \in \mathbb{P}$. Using (2_m), (3_m) we see that $\xi(t)s$, for every $\ell \leq m$, $n_0 < n_1 < \dots < n_\ell \leq m$, $t_i \in \mathbb{P}$ for $i \leq \ell + 2$, $\tau_i \in \psi_{n_i}^{(\mathcal{F})}(x_0, \dots, x_{n_{i-1}})$ for $i \leq \ell$ and $\tau \in \psi_{m+1}^{(\mathcal{F})}(x_0, \dots, x_m)$, if $\{f_{\tau_i}(t_i) : i \leq \ell\} \cup \{f_\tau(t)\}$ is a chain in \mathbb{P} with least element t_{\min} then the color of $\tau_0(x_{n_0}(t_0)) + \dots + \tau_\ell(x_{n_\ell}(t_\ell)) + \tau(s)$ is equal to the color of $\xi(t_{\min})$, and if $\{f_{\tau_i}(t_i) : i \leq \ell\} \cup \{f_\tau(t), t_{\ell+2}\}$ is a chain in \mathbb{P} with least element t_{\min} then the color of $\tau_0(x_{n_0}(t_0)) + \dots + \tau_\ell(x_{n_\ell}(t_\ell)) + \tau(s) + \xi(t_{\ell+2})$ is equal to the color of $\xi(t_{\min})$. This allows one to choose $x_{m+1}(t)$ for every $t \in \mathbb{P}$ in such a way that (1_{m+1}) and (2_{m+1}) are satisfied. This concludes the recursive construction. \square

3.4. Actions of trees on partial semigroups. Suppose that T is a finite rooted tree. As in Subsection 2.2, we consider T as an ordered set with respect to its canonical ordering. This is defined by setting $t_0 \leq t_1$ if and only if $t_0, t_1 \in T$ and t_0 is a descendent of t_1 .

Definition 3.3. Suppose that α is an action of a finite rooted tree T on an adequate partial semigroup S as in Definition 3.1. We say that α is Ramsey if, for every $\tau \in \mathcal{F}_\alpha$, the corresponding spine $f_\tau : T \rightarrow T$ is a regressive homomorphism, and for any finite subset S_0 of S , for any finite coloring c of S , and any finite subset \mathcal{F}_0 of \mathcal{F}_α , there exists a function $x : T \rightarrow S$ such that, for any $\tau \in \mathcal{F}_0$ and $t \in T$, $x(t) \in S_t \cap \varphi_S(S_0)$ and the color of $\tau(x(t))$ depends only on $f_\tau(t)$.

While the definition of adequate action might seem difficult to verify, it holds trivially in many examples, including the case of the action corresponding to Gowers' theorem for multiple tetris operations.

Theorem 3.4. Suppose that α is an action of a finite rooted tree on an adequate partial semigroup S given by some semigroup $\mathcal{F}_\alpha \subset \text{End}(S)$ such that, for every $\tau \in \mathcal{F}_\alpha$, the corresponding spine f_τ is a regressive homomorphism. The following statements are equivalent:

- (1) α is Ramsey;
- (2) the action induced by α on γS is Ramsey;
- (3) for any finite coloring c of S , sequence $(\psi_n^{(\mathcal{F})})$ of functions $\psi_n^{(\mathcal{F})} : (S^T)^n \rightarrow [\mathcal{F}_\alpha]^{<\aleph_0}$ and sequence $(\psi_n^{(S)})$ of functions $\psi_n^{(S)} : (S^T)^n \rightarrow [S]^{<\aleph_0}$ such that $\psi_n(x_0, \dots, x_{n-1})$ contains the range of $\tau_i \circ x_i$ for every $i \in n$ and $\tau_i \in \psi_i^{(\mathcal{F})}(x_0, \dots, x_{i-1})$, there exist functions $x_n : T \rightarrow S$ such that
 - $x_n(t) \in S_t \cap (\varphi_S \circ \psi_n^{(S)})(x_0, \dots, x_{n-1})$ for every $n \in \omega$ and $t \in T$; and
 - for any $\ell \in \omega$, $n_0 < n_1 < \dots < n_\ell \in \omega$, $t_i \in T$ for $i \leq \ell$, and $\tau_i \in \psi_{n_i}^{(\mathcal{F})}(x_0, \dots, x_{n_{i-1}})$ for $i \leq \ell$, if $\{f_{\tau_i}(t_i) : i \leq \ell\}$ is a chain in T with least element t , then the color of $\tau_0(x_{n_0}(t_0)) + \dots + \tau_\ell(x_{n_\ell}(t_\ell))$ depends only on t .

Proof. (1) \Rightarrow (2) Since α is Ramsey, we have that for any finite subset S_0 of S , for any finite coloring c of S , and any finite subset \mathcal{F}_0 of \mathcal{F}_α , there exists a function $x : T \rightarrow \varphi_S(S_0)$ such that, for any $\tau \in \mathcal{F}_0$ and $t \in T$, the color of $\tau(x(t))$ depends only on $f_\tau(t)$. By compactness of βS we deduce that there exists a function $\xi : T \rightarrow \gamma S$ such that, for any $\tau \in \mathcal{F}$, $t \in T$, and any finite coloring c of S , the color of $\tau(\xi(t))$ depends only on $f_\tau(t)$. This being true for any coloring of S implies that $\tau(\xi(t)) = \xi(f_\tau(t))$ for every $t \in T$ and $\tau \in \mathcal{F}_\alpha$. Therefore $\xi \in (\gamma S)_\alpha$.

(2) \Rightarrow (1) Suppose that $\xi \in (\gamma S)_\alpha$. Fix finite subsets S_0 of S , \mathcal{F}_0 of \mathcal{F}_α , and a finite coloring c of S . Consider the canonical extension of c to a finite coloring of βS . We have that, for every $t \in T$ and $\tau \in \mathcal{F}_0$, $\tau(\xi(t)) = \xi(f_\tau(t))$. In particular the color of $\tau(\xi(t))$ is equal to the color of $\xi(f_\tau(t))$. Fix $t \in T$. Using the notation of ultrafilter quantifiers, we have that $\xi(t)s, \forall \tau \in \mathcal{F}_0, s \in S_t \cap \varphi_S(S_0)$ and the color of $\tau(s)$ is equal to the color of $\xi(f_\tau(t))$. Therefore we can choose an element $x(t) \in S_t \cap \varphi_S(S_0)$ for every $t \in T$ such that the function $x : T \rightarrow S$ witnesses that the action α is Ramsey.

(3) \Rightarrow (1) Observe that the definition of Ramsey action is the particular instance of (4) where the sequence (x_n) has length 1.

(3) \Rightarrow (4) This is a consequence of Proposition 2.5 and Theorem 3.2. \square

In the Section 4 we will explain how various results in the literature can be seen as a special instance of Theorem 3.4.

3.5. Products of actions. We recall the notion of tensor product [16, Section 11.1]—see also [24, Section 1.2]—of ultrafilters. Suppose that \mathcal{U} and \mathcal{V} are ultrafilter on sets A, B , respectively. Then $\mathcal{U} \otimes \mathcal{V}$ is the ultrafilter on $A \times B$ obtained by declaring, for any $C \subset A \times B$, $C \in \mathcal{U} \otimes \mathcal{V}$ if and only if $(\mathcal{U}a), (\mathcal{V}b), (a, b) \in C$. Observe that in the particular case when \mathcal{U} is the principal ultrafilter over $a \in A$, then $C \in \mathcal{U} \otimes \mathcal{V}$ if and only if $\{b \in B : (a, b) \in C\} \in \mathcal{V}$. In the following we identify an element a of a set A with the corresponding principal ultrafilter. The following result is proved in [7, Corollary 2.8]

Theorem 3.5 (Bergelson–Hindman–Williams). *Suppose that $m, k \in \mathbb{N}$ and $\lambda : m \rightarrow k$ is a function. Suppose that for $i \in k$, $(S_i, +_i)$ is a semigroup and $p_i \in \gamma S_i$ is an idempotent element. Set $S := S_{\lambda(0)} \times \cdots \times S_{\lambda(m-1)}$ and suppose that A is a subset of S . If $A \in p_{\lambda(0)} \otimes \cdots \otimes p_{\lambda(m-1)}$ then for each $i \in k$ there exist a sequence $(y_{i,n})_{n \in \omega}$ in S_i such that*

$$\left\{ \left(\sum_{d \in F_s}^{\lambda(s)} y_{\lambda(s), d} \right)_{s \in m} : F_0 < F_1 < \cdots < F_{m-1} \text{ are finite subsets of } \omega \right\}$$

is contained in A .

In the statement of Theorem 3.5 and in the following, the expression $\sum_{d \in F}^j y_d$, where (y_n) is a sequence in a partial semigroup $(S, +_j)$, denotes the element $y_{d_0} +_j y_{d_1} +_j \cdots +_j y_{d_\ell}$ of S , where $\{d_0, \dots, d_\ell\}$ is an increasing enumeration of F . Whenever we write such an expression, we will also implicitly assert that $y_{d_0} +_j y_{d_1} +_j \cdots +_j y_{d_\ell}$ is defined in $(S, +_j)$.

We will now present a generalization of Theorem 3.5 to the setting of actions of ordered sets on adequate partial semigroups. Suppose that $m, k \in \mathbb{N}$ and $\lambda : m \rightarrow k$ is a function. For each $i \in k$ let $(S_i, +_i)$ be an adequate partial semigroup, and \mathbb{P}_i be a finite ordered set. For every $i \in k$ we can consider an action α_i of \mathbb{P}_i on $(S_i, +_i)$. This is given by a function $\mathbb{P}_i \rightarrow \mathcal{S}(S_i, +_i)$, $t \mapsto S_{i,t}$ such that $S_{i,t} \leq S_{i,t'}$ whenever $t \leq t'$, and a semigroup \mathcal{F}_{α_i} of adequate partial semigroup homomorphisms of $(S_i, +_i)$ such that for every $\tau \in \mathcal{F}_{\alpha_i}$ there exists a function $f_\tau : \mathbb{P}_i \rightarrow \mathbb{P}_i$ with the property that τ maps $S_{i,t}$ to $S_{i,f_\tau(t)}$ for every $t \in \mathbb{P}_i$. Recall that each action α_i extends to an action of \mathbb{P}_i on γS_i . In this case $(\gamma S_i)_{\alpha_i}$ denotes the set of functions $\xi : \mathbb{P}_i \rightarrow \gamma S_i$ such that, for any $t, t' \in \mathbb{P}_i$, $\xi(t)$ is an idempotent element of $\gamma S_{i,t}$ and $\xi(t) \leq \xi(t')$ if $t \leq t'$. The following result is the natural generalization of Theorem 3.5 to the setting of actions of ordered sets on partial semigroups.

Theorem 3.6. *Suppose that $\xi_i \in (\gamma S_i)_{\alpha_i}$ is an order-preserving idempotent for $i \in k$. Set $S := S_{\lambda(0)} \times \cdots \times S_{\lambda(m-1)}$ and suppose that c is a finite coloring of S . Extend c canonically to a coloring of βS . Fix for $i \in k$ a sequence $(\psi_{i,n}^{(\mathcal{F})})$ of functions $\psi_{i,n}^{(\mathcal{F})} : (S_i^{\mathbb{P}_i})^n \rightarrow [\mathcal{F}_{\alpha_i}]^{<\aleph_0}$ and a sequence $(\psi_{i,n}^{(S)})$ of functions $\psi_{i,n}^{(S)} : (S_i^{\mathbb{P}_i})^n \rightarrow [S_i]^{<\aleph_0}$ such that $\psi_{i,n}(x_{i,0}, \dots, x_{i,n-1})$ contains the range of $\tau_j \circ x_j$ for every $j \in n$ and $\tau_j \in \psi_{i,j}^{(\mathcal{F})}(x_{i,0}, \dots, x_{i,j-1})$. Then there exist sequences $(x_{i,n})_{n \in \omega}$ of functions $x_{i,n} : \mathbb{P}_i \rightarrow S_i$ for $i \in k$ such that*

- $x_{i,n}(t) \in S_{i,t} \cap (\varphi_{S_i} \circ \psi_{i,n}^{(S)})(x_{i,0}, \dots, x_{i,n-1})$ for every $n \in \omega$, $i \in k$, and $t \in \mathbb{P}_i$; and
- for any $\ell \in \omega$, $i \in k$, finite subsets $F_0 < F_1 < \cdots < F_{\ell-1}$ of ω , sequences $(t_{i,n})_{n \in \omega}$ in \mathbb{P}_i and $(\tau_{i,n})_{n \in \omega}$ in \mathcal{F}_{α_i} such that $\tau_{i,n} \in \psi_{i,n}^{(\mathcal{F})}(x_{i,0}, \dots, x_{i,n-1})$ for every $n \in \omega$, if $\{f_{\tau_{\lambda(s),d}}(t_{\lambda(s),d}) : d \in F_s\}$ is a chain in $\mathbb{P}_{\lambda(s)}$ with least element t_s for every $s \in m$, then the color of

$$\left(\sum_{d \in F_s}^{\lambda(s)} \tau_{\lambda(s),d}(x_{\lambda(s),d}(t_{\lambda(s),d})) \right)_{s \in m}$$

is the same as the color of $\xi_{\lambda(0)}(t_0) \otimes \xi_{\lambda(1)}(t_1) \otimes \cdots \otimes \xi_{\lambda(m-1)}(t_{m-1})$.

Proof. We define by recursion a $\ell \in \omega$ functions $x_{i,\ell} : \mathbb{P}_i \rightarrow S_i$ with $x_{i,\ell}(t) \in S_{i,t} \cap (\varphi_{S_i} \circ \psi_{i,\ell}^{(S)})(x_{i,0}, \dots, x_{i,\ell-1})$ for $i \in k$ such that for every $\ell \in \omega$ the following holds:

- (1 $_\ell$) for every $j \in m$, finite subsets $F_0 < F_1 < \dots < F_j < \ell$ of ω , sequences $(t_{i,d})_{d \in \ell}$ in \mathbb{P}_i and $(\tau_{i,d})_{d \in \ell}$ in \mathcal{F}_{α_i} such that $\tau_{i,d} \in \psi_{i,d}^{(\mathcal{F})}(x_0, \dots, x_{d-1})$ for every $d \in \ell$, and $h_s \in \mathbb{P}_{\lambda(s)}$ for $j < s < m$, if $\{f_{\tau_{\lambda(s),d}}(t_{\lambda(s),d}) : d \in F_s\}$ is a chain in $\mathbb{P}_{\lambda(s)}$ with least element t_s for every $s \leq j$, then the color of

$$\left(\sum_{d \in F_0}^{\lambda(0)} \tau_{\lambda(0),d}(x_{\lambda(0),d}(t_0)) \right) \otimes \dots \otimes \left(\sum_{d \in F_j}^{\lambda(j)} \tau_{\lambda(j),d}(x_{\lambda(j),d}(t_j)) \right) \otimes \xi_{\lambda(j+1)}(h_{j+1}) \otimes \dots \otimes \xi_{\lambda(m-1)}(h_{m-1})$$

(where we identify an element of S_i with the corresponding principal ultrafilter) is the same as the color of

$$\xi_{\lambda(0)}(t_0) \otimes \xi_{\lambda(1)}(t_1) \otimes \dots \otimes \xi_{\lambda(j)}(t_j) \otimes \xi_{\lambda(j+1)}(h_{j+1}) \otimes \dots \otimes \xi_{\lambda(m-1)}(h_{m-1}),$$

and

- (2 $_\ell$) for every $j \in m$, finite subsets $F_0 < F_1 < \dots < F_j < \ell$, sequences $(t_{i,d})_{d \leq \ell}$ in \mathbb{P}_i and $(\tau_{i,d})_{d \in \ell}$ in \mathcal{F}_{α_i} such that $\tau_{i,d} \in \psi_{i,d}^{(\mathcal{F})}(x_0, \dots, x_{d-1})$ for every $d \in \ell$, and $h_s \in \mathbb{P}_{\lambda(s)}$ for $j < s < m$, if $\{f_{\tau_{\lambda(s),d}}(t_{\lambda(s),d}) : d \in F_s\}$ is a chain in $\mathbb{P}_{\lambda(s)}$ with least element t_s for every $s \in j$ and $\{f_{\tau_{\lambda(j),d}}(t_{\lambda(j),d}) : d \in F_j\} \cup \{t_{\lambda(j),\ell}\}$ is a chain in $\mathbb{P}_{\lambda(j)}$ with least element t_j , then the color of

$$\left(\sum_{d \in F_0}^{\lambda(0)} \tau_{\lambda(0),d}(x_{\lambda(0),d}(t_{\lambda(0),d})) \right) \otimes \dots \otimes \left(\sum_{d \in F_{j-1}}^{\lambda(j-1)} \tau_{\lambda(j-1),d}(x_{\lambda(j-1),d}(t_{\lambda(j-1),d})) \right) \otimes$$

$$\otimes \left(\sum_{d \in F_j}^{\lambda(j)} \tau_{\lambda(j),d}(x_{\lambda(j),d}(t_{\lambda(j),d})) +_{\lambda(j)} \xi_{\lambda(j)}(t_{\lambda(j),\ell}) \right) \otimes \xi_{\lambda(j+1)}(h_{j+1}) \otimes \dots \otimes \xi_{\lambda(m-1)}(h_{m-1})$$

is the same as the color of

$$\xi_{\lambda(0)}(t_0) \otimes \xi_{\lambda(1)}(t_1) \otimes \dots \otimes \xi_{\lambda(j)}(t_j) \otimes \xi_{\lambda(j+1)}(h_{j+1}) \otimes \dots \otimes \xi_{\lambda(m-1)}(h_{m-1}),$$

Suppose that such a sequence has been defined up to ℓ in such a way that (1 $_\ell$) and (2 $_\ell$) are satisfied. From (2 $_\ell$) and the fact that ξ_i for $i \in k$ is an order-preserving idempotent in $(\gamma S_i)_{\alpha_i}$, it follows that the following holds as well:

- (3 $_\ell$) for every $j \in m$, finite subsets $F_0 < F_1 < \dots < F_j < \ell$, sequences $(t_{i,d})_{d \leq \ell+1}$ in \mathbb{P}_i and $(\tau_{i,d})_{d \in \ell}$ in \mathcal{F}_{α_i} such that $\tau_{i,d} \in \psi_{i,d}^{(\mathcal{F})}(x_0, \dots, x_{d-1})$ for every $d \leq \ell$, and $h_s \in \mathbb{P}_{\lambda(s)}$ for $j < s < m$, if $\{f_{\tau_{\lambda(s),d}}(t_{\lambda(s),d}) : d \in F_s\}$ is a chain in $\mathbb{P}_{\lambda(s)}$ with least element t_s for every $s \in j$, and $\{f_{\tau_{\lambda(j),d}}(t_{\lambda(j),d}) : d \in F_j\} \cup \{t_{\lambda(j),\ell}, t_{\lambda(j),\ell+1}\}$ is a chain in $\mathbb{P}_{\lambda(j)}$ with least element t_j , then the color of

$$\left(\sum_{d \in F_0}^{\lambda(0)} \tau_{\lambda(0),d}(x_{\lambda(0),d}(t_{\lambda(0),d})) \right) \otimes \dots \otimes \sum_{d \in F_{j-1}}^{\lambda(j-1)} \tau_{\lambda(j-1),d}(x_{\lambda(j-1),d}(t_{\lambda(j-1),d})) \otimes$$

$$\otimes \left(\sum_{d \in F_j}^{\lambda(j)} \tau_{\lambda(j),d}(x_{\lambda(j),d}(t_{\lambda(j),d})) +_{\lambda(j)} \xi_{\lambda(j)}(t_{\lambda(j),\ell}) +_{\lambda(j)} \xi_{\lambda(j)}(t_{\lambda(j),\ell+1}) \right) \otimes \xi_{\lambda(j+1)}(h_{j+1}) \otimes \dots \otimes \xi_{\lambda(m-1)}(h_{m-1})$$

is the same as the color of

$$\xi_{\lambda(0)}(t_0) \otimes \xi_{\lambda(1)}(t_1) \otimes \dots \otimes \xi_{\lambda(j)}(t_j) \otimes \xi_{\lambda(j+1)}(h_{j+1}) \otimes \dots \otimes \xi_{\lambda(m-1)}(h_{m-1}),$$

Considering the definition of the operations between ultrafilters, one can then see that, for every $i \in k$ and $t \in \mathbb{P}_i$,

- if follows from (2 $_m$) that the set of possible choices of $x_{i,\ell+1}(t) \in S_{i,t}$ satisfying (1 $_{\ell+1}$) whenever $s \in m$, $\lambda(s) = i$, and $t_{\lambda(s),\ell} = t$, belongs to $\xi_i(t)$, and
- it follows from (3 $_m$) that the set of possible choices of $x_{i,\ell+1}(t) \in S_{i,t}$ satisfying (2 $_{\ell+1}$) whenever $s \in m$, $\lambda(s) = i$, and $t_{\lambda(s),\ell} = t$ belongs to $\xi_i(t)$.

Therefore one can choose $x_{i,m+1} : \mathbb{P} \rightarrow S_i$ with $x_{i,m+1}(t) \in S_{i,t} \cap (\varphi_{S_i} \circ \psi_{i,m+1}^{(S)})(x_{i,0}, \dots, x_{i,m})$ for $i \in k$ such that both (1 $_{m+1}$) and (2 $_{m+1}$) are satisfied for every $t \in \mathbb{P}$. This concludes the recursive construction. \square

Corollary 3.7. Suppose that T_i for $i \in k$ are finite rooted trees, and α_i for $i \in k$ is a Ramsey action of T_i on the adequate partial semigroup $(S_i, +_i)$. Set $S := S_{\lambda(0)} \times \dots \times S_{\lambda(m-1)}$ and suppose that c is a finite coloring of S . Fix for $i \in k$ a sequence $(\psi_{i,n}^{(\mathcal{F})})$ of functions $\psi_{i,n}^{(\mathcal{F})} : (S_i^{T_i})^n \rightarrow [\mathcal{F}_{\alpha_i}]^{<\aleph_0}$ and a sequence $(\psi_{i,n}^{(S)})$ of functions $\psi_{i,n}^{(S)} : (S_i^{T_i})^n \rightarrow [S_i]^{<\aleph_0}$ such that $\psi_{i,n}(x_{i,0}, \dots, x_{i,n-1})$ contains the range of $\tau_j \circ x_j$ for every $j \in n$ and $\tau_j \in \psi_{i,j}^{(\mathcal{F})}(x_{i,0}, \dots, x_{i,j-1})$. Then there exist sequences $(x_{i,n})_{n \in \omega}$ of functions $x_{i,n} : T_i \rightarrow S_i$ such that

- $x_{i,n}(t) \in S_{i,t} \cap (\varphi_{S_i} \circ \psi_{i,n}^{(S)})(x_{i,0}, \dots, x_{i,n-1})$ for every $n \in \omega$, $i \in k$, and $t \in T_i$; and
- for any $i \in k$, finite subsets $F_0 < F_1 < \dots < F_{m-1}$ of ω , sequences $(t_{i,n})_{n \in \omega}$ in T_i and $(\tau_{i,n})_{n \in \omega}$ in \mathcal{F}_{α_i} such that $\tau_{i,n} \in \psi_{i,n}^{(\mathcal{F})}(x_0, \dots, x_{n-1})$ for every $n \in \omega$, if $\{f_{\tau_{\lambda(s),d}}(t_{i,d}) : d \in F_{\lambda(s)}\}$ is a chain in $T_{\lambda(s)}$ with least element t_s for every $s \in m$, then the color of

$$\left(\sum_{d \in F_i}^{\lambda(s)} \tau_{\lambda(i),d}(x_{\lambda(i),d}(t_{\lambda(i),d})) \right)_{s \in m}$$

depends only on $(t_s)_{s \in m}$.

4. EXAMPLES OF ACTIONS OF TREES ON PARTIAL SEMIGROUPS

4.1. Gowers' theorem for multiple tetris operations. Suppose that T is a finite rooted tree with root r , and let T^+ be the set of nodes of T different from the root. We regard as above T as an ordered set with respect to its canonical tree ordering. Let FIN^T be the space of functions $b : \text{dom}(b) \rightarrow T^+$ where $\text{dom}(b)$ is a finite (possibly empty) subset of ω and the range of b is contained in a branch of T . Then FIN^T is a partial semigroup with respect of the partially defined operation $(b_0, b_1) \mapsto b_0 + b_1$, which is defined whenever the intersection of the domains of b_0, b_1 is empty. In this case, $b_0 + b_1$ is defined to be the union of b_0 and b_1 (where we identify functions with their graph).

There is a natural action of T on FIN^T defined as follows. For every $t \in T^+$ let FIN_t^T be the space of $b \in \text{FIN}^T$ such that the range of b is a chain in T with least element t . We also let FIN_r^T be the set containing only the empty function. Any regressive homomorphism $f : T \rightarrow T$ defines a canonical adequate partial semigroup homomorphism $\tau_f : \text{FIN}^T \rightarrow \text{FIN}^T$ as follows. Suppose that $b \in \text{FIN}^T$. Then $\tau_f(b)$ has domain $\{n \in \text{dom}(b) : f(b(n)) \in T^+\}$ and it is defined by $\tau_f(b) : n \mapsto f(b(n))$. Observe that τ_f maps FIN_t^T to $\text{FIN}_{f(t)}^T$ for every $t \in T$. Therefore τ_f has *spine* f according to Definition 3.1. The collection \mathcal{F}_α of adequate partial semigroups homomorphisms τ_f where f varies among all the regressive homomorphisms $f : T \rightarrow T$ is an action of T on the adequate partial semigroup FIN^T according to Definition 3.1. It is easily seen that such an action is Ramsey as in Definition 3.3. Indeed consider a finite subset S_0 of FIN^T and a finite coloring c of FIN^T . Fix any nonempty finite subset A of ω disjoint from the union of the supports of the elements of S_0 . Consider then the function $x : T \rightarrow \text{FIN}^T$, $t \mapsto b_t$ defined by setting $b_t \in \text{FIN}_t^T$ to be the function with domain A and constantly equal to t . This witnesses that such an action is Ramsey. From Theorem 3.4 one can deduce the following.

Theorem 4.1. *Suppose that T is a finite rooted tree. For any finite coloring c of FIN^T , for any sequence (d_n) of elements of ω , there exists sequences $(b_{t,n})_{n \in \omega}$ for $t \in T$ of elements $b_{t,n} \in \text{FIN}_t^T$ such that*

- $\text{dom}(b_{t,n}) > d_n$ and $\text{dom}(b_{t,n+1}) > \text{dom}(b_{t,n})$ for every $t \in T$ and $n \in \omega$;
- for any $\ell \in \omega$, $n_0 < n_1 < \dots < n_\ell \in \omega$, $t_0, \dots, t_\ell \in T$, and regressive homomorphisms $f_i : T \rightarrow T$ for $i \leq \ell$, if $\{f_i(t_i) : i \leq \ell\}$ is a chain in T with least element t , then the color of $\tau_{f_0}(b_{t_0,n_0}) + \dots + \tau_{f_\ell}(b_{t_\ell,n_\ell})$ depends only on t .

From Corollary 3.7 one can deduce the following generalization of Theorem 4.1.

Theorem 4.2. *Suppose that $m, k \in \mathbb{N}$ and $\lambda : m \rightarrow k$ is a function. Let T_i for $i \in k$ be a finite rooted tree. Fix any coloring c of $\text{FIN}^{T_{\lambda(0)}} \times \dots \times \text{FIN}^{T_{\lambda(m-1)}}$. For any sequence (d_n) of elements of ω , there exist sequences $(b_{i,t,n})_{n \in \omega}$ in $\text{FIN}_t^{T_i}$ for $i \in k$ and $t \in T_i$ such that*

- $\text{dom}(b_{i,t,n}) > d_n$ and $\text{dom}(b_{i,t,n+1}) > \text{dom}(b_{i,t,n})$ for every $i \in k$, $t \in T_i$, and $n \in \omega$;
- for finite subsets $F_0 < F_1 < \dots < F_{m-1}$ of ω , nodes $t_{i,d}$ of T_i and regressive homomorphisms $f_{i,d}$ of T_i for $d \in \omega$ such that $\{f_{\lambda(s),d}(t_{\lambda(s),d}) : d \in F_s\}$ is a chain in $T_{\lambda(s)}$ with least element t_s for every $s \in m$, the color of

$$\left(\sum_{d \in F_s} \tau_{f_{\lambda(s),d}}(b_{f(s),t_{\lambda(s),d}}) \right)_{s \in m}$$

depends only on $(t_s)_{s \in m}$.

Gowers' theorem for multiple tetris operations [18, Theorem 1.1] is the particular instance of Theorem 3.4 where T is the rooted tree I_k with $k+1$ nodes such that the canonical ordering on I_k is a linear order. In other words, I_k is just the tree whose nodes are the initial segments of k ordered by reverse inclusion. The original version of Gowers' theorem [12] corresponds to the case when the only regressive homomorphisms of I_k considered are the ones mapping a node its j -predecessor for some $j \geq 0$ (with the convention that the j -th predecessor of a node of height at most j is equal to the root).

4.2. The Hales-Jewett theorem for located words. Let L be a set (*alphabet*) and v a symbol not in L (*variable*). A *located word* in the alphabet L is a finitely supported function $b : \text{dom}(b) \rightarrow L$ where $\text{dom}(b)$ is a (possibly empty) finite subset of ω . Similarly, a *located variable word* in the alphabet L and variable v is a finitely supported function $b : \text{dom}(b) \rightarrow L \cup \{v\}$ whose range contains v , where $\text{dom}(b)$ is a finite subset of ω . Following [24, Section 2.6], we let FIN_L be the set of located words in L , and FIN_{Lv} be the set of located variable words in L and the variable v . Then $S := \text{FIN}_L \cup \text{FIN}_{Lv}$ has a natural partial semigroup operation, obtained by letting $b_0 + b_1$ be defined whenever the domains of b_0 and b_1 are disjoint. In such a case, $b_0 + b_1$ is just $b_0 \cup b_1$ (where we identify a function with its graph). It is clear that FIN_L is an adequate subsemigroup of S , while FIN_{Lv} is an adequate ideal of S .

Let I_1 be the rooted tree with only two nodes (including the root). We denote the root of I_1 by r , and the other node by t . Every element a of L defines an adequate semigroup homomorphism $\tau_a : S \rightarrow S$ obtained by replacing every occurrence of the variable v with a . This defines an action α of I_1 on S , where the closed subsemigroup corresponding to the root r of I_1 is FIN_L , and the closed subsemigroup corresponding to the other node t of I_1 is FIN_{Lv} . In order to see that such an action is Ramsey in the sense of Definition 3.3, consider any minimal idempotent p_r of γFIN_L . Then by Lemma 2.1 there exists a minimal idempotent p_t of γFIN_{Lv} such that $p_t \leq p_r$. By minimality of p_r we deduce that $\tau_a(p_t) = p_r$ for every $a \in L$. Therefore the pair (p_r, p_t) witnesses that $(\gamma S)_\alpha$ is nonempty by Theorem 3.4.

The infinite Hales-Jewett theorem for located words [4, Theorem 6.1]—see also [24, Theorem 2.40]—is then a consequence of Theorem 3.4 applied to such a Ramsey action α . From Corollary 3.7 one can deduce the following version.

Theorem 4.3. *Suppose that $m, k \in \mathbb{N}$ and $\lambda : m \rightarrow k$ is a function. Let L_i for $i \in k$ be a set. Fix a coloring of $\text{FIN}_{L_{\lambda(0)}} \times \cdots \times \text{FIN}_{L_{\lambda(m-1)}}$. For any sequence (d_n) of elements of ω , and sequences $(L_{i,n})_{n \in \omega}$ of finite subsets of L_i for $i \in k$, there exists sequences $(b_{i,n})_{n \in \omega}$ in $\text{FIN}_{L_{i,v}}$ for $i \in k$ such that*

- $\text{dom}(b_{i,n}) > d_n$ and $\text{dom}(b_{i,n+1}) > \text{dom}(b_{i,n})$ for every $i \in k$ and $n \in \omega$;
- the set of elements of $\text{FIN}_{L_{\lambda(0)}} \times \cdots \times \text{FIN}_{L_{\lambda(m-1)}}$ of the form

$$\left(\sum_{d \in F_s} \tau_{a_{f(s),d}}(b_{f(s),d}) \right)_{s \in m}$$

for $F_0 < F_1 < \cdots < F_{m-1}$ and $a_{i,d} \in L_i$ for $d \in \omega$ and $i \in k$ is monochromatic.

More generally, one can regard any *layered action* on a partial semigroup S in the sense of [10, Definition 3.3] as an action of the tree I_k on S in the sense of Definition 3.1. Any such an action is automatically Ramsey, although this is not easy to see directly. Rather, it follows from [10, Lemma 2.19] and Proposition 2.7 or, alternatively, [10, Theorem 3.8]. Once one observe that any layered action is a Ramsey action of I_k , one can see that Theorem 3.4 subsumes [10, Theorem 3.13].

4.3. A common generalization. Let L be a set (alphabet). Suppose that T is a finite rooted tree with root r , and T^+ is the set of nodes of T different from the root. We regard the nodes of T^+ as *variables*. We let $\text{FIN}_{L,r}^T$ be the set of functions $b : \text{dom}(b) \rightarrow L$, where $\text{dom}(b)$ is a (possibly empty) subset of ω . For $t \in T^+$ let $\text{FIN}_{L,t}^T$ be the set of functions $b : \text{dom}(b) \rightarrow L \cup T^+$ such that $\text{dom}(b)$ is a finite subset of ω , and the intersection of the range of b with T^+ is a nonempty chain with least element t . Let FIN_L^T be the set of functions $b : \text{dom}(b) \rightarrow L \cup T^+$, where $\text{dom}(b)$ is a (possibly empty) finite subset of ω . Then FIN_L^T is endowed with a natural partial semigroup operation. If $b_0, b_1 \in \text{FIN}_L^T$, then $b_0 + b_1$ is defined if and only if the domains of b_0 and b_1 are disjoint. In such a case one has that $b_0 + b_1$ is the union of b_0 and b_1 .

Definition 4.4. A *variable substitution map* is a *partially defined* function σ from a subset $\text{dom}(\sigma)$ of $T^+ \cup L$ to $T^+ \cup L$ such that

- $L \subset \text{dom}(\sigma)$ and $\sigma|_L$ is the identity function of L , and
- the function $f_\sigma : T \rightarrow T$ defined by

$$t \mapsto \begin{cases} \sigma(t) & \text{if } t \in \text{dom}(\sigma) \text{ and } \sigma(t) \in T^+, \\ r & \text{otherwise} \end{cases}$$

is a regressive homomorphism of T .

Any variable substitution map σ defines an adequate partial semigroup homomorphism τ_σ of FIN_L^T as follows. Suppose that $b \in \text{FIN}_L^T$. Then the domain of $\tau_\sigma(b)$ is $\{n \in \text{dom}(b) : b(n) \in \text{dom}(\sigma)\}$, and $\tau_\sigma(b)$ is the function

$n \mapsto \sigma(b(n))$. The collection of such maps defines an action of T on FIN_L^T in the sense of Definition 3.1. The spine of the variable substitution map τ_σ is the regressive homomorphism $f_\sigma : T \rightarrow T$.

We claim that such an action is Ramsey; see Definition 3.3 and Theorem 3.4. Fix any minimal idempotent element $\xi(r)$ of FIN_L . By Lemma 2.1 for every $t_0 \in T$ of height 1 there exists a minimal idempotent element $\xi(t_0)$ of $\gamma\text{FIN}_{L,t_0}^T$ such that $\xi(t_0) \leq \xi(r)$. By minimality of $\xi(r)$, if σ is any variable substitution map such that $f_\sigma(t_0) = r$ one has that $\tau_\sigma(\xi(t_0)) = \xi(r)$. Fix now $t \in T^+$ and consider a predecessor t_0 of T of height 1. Let $\psi_t : \text{FIN}_{L,t_0}^T \rightarrow \text{FIN}_{L,t}^T$ be the function defined by letting, for $b \in \text{FIN}_{L,t_0}^T$, $\psi_t(b)$ be the function with the same domain as b such that

$$\psi_t(b)(n) = \begin{cases} t & \text{if } b(n) = t_0 \\ b(n) & \text{otherwise.} \end{cases}$$

Since ψ_t is an adequate semigroup homomorphism, it extends to a continuous semigroup homomorphism $\psi_t : \gamma\text{FIN}_{L,t_0}^T \rightarrow \gamma\text{FIN}_{L,t}^T$. Define now $\xi(t) := \psi_t(\xi(t_0))$. Observe that, if σ is a variable substitution map, then $\tau_\sigma(\xi(t)) = \xi(f_\sigma(t))$. Indeed, if $f_\sigma(t) = s \in T^+$ then we have that $\tau_\sigma(\xi(t)) = (\tau_\sigma \circ \psi_t)(\xi(t_0)) = \psi_s(\xi(t_0)) = \xi(s)$. If $f_\sigma(t) = r$ then fix any variable substitution map σ' such that $\sigma'(t_0) = \sigma(t)$ (where the equality should be interpreted as asserting that the left hand side is not defined if the right hand side is not defined). Then we have that $\tau_\sigma(\xi(t)) = (\tau_\sigma \circ \psi_t)(\xi(t_0)) = \tau_{\sigma'}(\xi(t_0)) = \xi(r) = \xi(f_\sigma(t))$. This concludes the proof.

Therefore applying Theorem 3.4 one obtains the following result, which is a common generalization of Gowers' theorem for multiple tetris operations and the infinite Hales–Jewett theorem for located words [4, Theorem 6.1].

Theorem 4.5. *Suppose that T is a finite tree, L is a set, and c is a finite coloring of FIN_L^T . For any sequence (d_n) in ω and sequence (\mathcal{F}_n) where \mathcal{F}_n is a finite set of variable substitution maps for T, L as in Definition 4.4, there exist sequences $(b_{t,n})_{n \in \omega}$ in $\text{FIN}_{L,t}^T$ for $t \in T$ such that*

- for every $n \in \omega$ and $t \in T$, $\text{dom}(b_{t,n}) > d_n$ and $\text{dom}(b_{t,n+1}) > \text{dom}(b_{t,n})$;
- for every $\ell \in \omega$, $n_0 < \dots < n_\ell \in \omega$, $t_0, \dots, t_\ell \in T$, variable substitution maps $\sigma_0, \dots, \sigma_\ell$, such that $\{f_{\sigma_0}(t_0), \dots, f_{\sigma_\ell}(t_\ell)\}$ is a chain in T least element t , one has that the color of

$$\tau_{\sigma_0}(b_{t_0,n_0}) + \dots + \tau_{\sigma_\ell}(b_{t_\ell,n_\ell})$$

depends only on t .

Theorem 3.4 is the particular instance of Theorem 4.5 where $L = \emptyset$. The infinite Hales–Jewett theorem for located words [4, Theorem 6.1]—see also [24, Theorem 2.40]—is the particular instance of Theorem 4.5 where T is the rooted tree with only two nodes.

More generally from Corollary 3.7 one can deduce the following generalization of Theorem 4.5.

Theorem 4.6. *Suppose that $m, k \in \mathbb{N}$ and $\lambda : m \rightarrow k$ is a function. Let T_i be a finite rooted tree and L_i be a set for $i \in k$. Fix any coloring c of $\text{FIN}_{L_{\lambda(0)}}^{T_{\lambda(0)}} \times \dots \times \text{FIN}_{L_{\lambda(m-1)}}^{T_{\lambda(m-1)}}$. For any sequence (d_n) of elements of ω , $i \in k$, sequences $(\mathcal{F}_{i,n})$ where $\mathcal{F}_{i,n}$ is a finite set of variable substitution maps for T_i, L_i as in Definition 4.4, for every $i \in k$ and $t \in T_i$, there exist sequences $(b_{i,t,n})_{n \in \omega}$ in $\text{FIN}_{L_i,t}^{T_i}$ such that*

- the least element of the domain of $b_{i,t,n}$ is larger than d_n and larger than the largest element of the domain of $b_{i,t,n-1}$, and
- for any $F_0 < F_1 < \dots < F_{m-1}$, nodes $t_{i,d}$ of T_i , variable substitution maps $\sigma_{i,d} \in \mathcal{F}_{i,d}$ for $d \in \omega$ and $i \in k$ such that $\{f_{\sigma_{\lambda(s),d}}(t_{\lambda(s),d}) : d \in F_s\}$ is a chain in $T_{\lambda(s)}$ with least element t_s for every $s \in m$, the color of

$$\left(\sum_{d \in F_s} \tau_{\sigma_{\lambda(s),d}}(b_{f(s),t_{\lambda(s),d}}) \right)_{s \in m}$$

depends only on $(t_s)_{s \in m}$.

5. ACTIONS OF TREES ON FILTERED SEMIGROUPS

5.1. Filtered sets and filtered semigroup. By a *filtered set* we mean a set S endowed with a distinguished filter \mathfrak{F} . An ultrafilter \mathcal{U} on the filtered set S is *cofinal* if it contains \mathfrak{F} . We then let $\beta_{\mathfrak{F}}S$ be the closed subsets of βS consisting of the cofinal ultrafilters on S . Given a filter \mathfrak{F} on S we define the corresponding *dual coideal* to be the collection \mathfrak{F}^* of subsets A of S whose complement does not belong to \mathfrak{F} (equivalently, $A \cap C \neq \emptyset$ for every $C \in \mathfrak{F}$); see [5, Section 1] and also [1, Section 2]. It is not difficult to verify that this is indeed a coideal in the sense of [24, Section 1.1]. Observe that an ultrafilter \mathcal{U} over S contains \mathfrak{F} if and only if it is contained in \mathfrak{F}^* . Furthermore, \mathfrak{F}^* is the union of all the ultrafilters in $\beta_{\mathfrak{F}}S$, while \mathfrak{F} is the intersection of all the ultrafilters in $\beta_{\mathfrak{F}}S$; see [16, Theorem 3.11].

Suppose now that $(S, +)$ is a partial semigroup. Let C be a subset of S and a be an element of S . Using notation from [10] we denote by $-a + C$ the set of elements b of S such that $a + b$ is defined and belongs to C .

Definition 5.1. A filtered semigroup is a triple $(S, \mathfrak{F}, +)$ where $(S, +)$ is a partial semigroup, \mathfrak{F} is a filter on S such that

- for every $x \in S$, \mathfrak{F} contains $\{y \in S : x + y \text{ is defined}\}$;
- for any $C \in \mathfrak{F}$ and for any $B \in \mathfrak{F}^*$ there exists a finite subset F of B such that $\bigcup_{a \in F} (-a + C) \in \mathfrak{F}$.

Proposition 5.2. Suppose that $(S, +)$ is an adequate partial semigroup, and \mathfrak{F} is a filter on S . Then $(S, \mathfrak{F}, +)$ is a filtered semigroup if and only if $\beta_{\mathfrak{F}}S$ is a closed subsemigroup of γS .

Proof. Suppose that $(S, \mathfrak{F}, +)$ is a filtered semigroup. Since \mathfrak{F} contains $\{y \in S : x + y \text{ is defined}\}$ for every $x \in S$, it follows that $\beta_{\mathfrak{F}}S$ is contained in γS . We want to prove that $\beta_{\mathfrak{F}}S$ is a subsemigroup of γS . Let \mathcal{U}, \mathcal{V} be ultrafilters in $\beta_{\mathfrak{F}}S$. Suppose by contradiction that $\mathcal{U} + \mathcal{V} \notin \beta_{\mathfrak{F}}S$. Then there exists $C \in \mathfrak{F}$ such that $S \setminus C \in \mathcal{U} + \mathcal{V}$. Therefore we have that $B := \{x \in S : \forall y, x + y \in S \setminus C\} \in \mathcal{U} \subset \mathfrak{F}^*$. Suppose that F is a finite subset of B . Then we have that $\forall y, \forall a \in F, a + y \in S \setminus C$. Therefore $\{y \in Y : \forall a \in F, a + y \in S \setminus C\} \in \mathcal{V} \subset \mathfrak{F}^*$. Therefore $\{y \in Y : \exists a \in F, a + y \in C\} \notin \mathfrak{F}$. This contradicts the assumption that $(S, \mathfrak{F}, +)$ is a filtered semigroup.

Conversely, suppose that $\beta_{\mathfrak{F}}S$ is a subsemigroup of γS . Thus we have that

$$\mathfrak{F} = \bigcap \beta_{\mathfrak{F}}S \supset \bigcap \gamma S \supset \{\{y \in S : x + y \text{ is defined}\} : x \in S\}.$$

This shows that \mathfrak{F} satisfies the first condition in the definition of filtered semigroup. We now want to verify the second condition. Suppose by contradiction that there exist $C \in \mathfrak{F}$ and $B \in \mathfrak{F}^*$ such that for any finite subset F of B one has that $\bigcup_{a \in F} (-a + C) \notin \mathfrak{F}$. Therefore we have that $\bigcap_{a \in F} (-a + S \setminus C) \in \mathfrak{F}^*$ for every finite subset F of B . Therefore by [16, Theorem 3.11] there exists an ultrafilter \mathcal{V} over S such that $\{-a + S \setminus C : a \in B\} \subset \mathcal{V} \subset \mathfrak{F}^*$, as well as an ultrafilter \mathcal{U} over S such that $B \in \mathcal{U} \subset \mathfrak{F}^*$. By the choice of \mathcal{U} and \mathcal{V} we have that $C \notin \mathcal{U} + \mathcal{V}$. Therefore \mathcal{U}, \mathcal{V} are elements of $\beta_{\mathfrak{F}}S$ such that $\mathcal{U} + \mathcal{V} \notin \beta_{\mathfrak{F}}S$, contradicting our assumption that $\beta_{\mathfrak{F}}S$ is a subsemigroup of γS . \square

Theorem 4.28 of [16] is the particular instance of Proposition 5.2 in the case when the operation in S is everywhere defined and \mathfrak{F} is the filter of cofinite subsets of S .

Suppose that (S, \mathfrak{F}) and (T, \mathfrak{G}) are filtered semigroups, and $\sigma : S \rightarrow T$ is a function. We say that σ is a *filtered semigroup homomorphism* if σ is a partial semigroup homomorphism and $\sigma(\mathfrak{F}) = \mathfrak{G}$, where $\sigma(\mathfrak{G})$ is the collection of subsets of T whose inverse image under σ belongs to \mathfrak{F} . This implies that the canonical continuous extension $\sigma : \beta S \rightarrow \beta T$ restricts to a continuous semigroup homomorphism from $\beta_{\mathfrak{F}}S$ to $\beta_{\mathfrak{G}}T$. We say that (T, \mathfrak{G}) is a *filtered subsemigroup* of (S, \mathfrak{F}) if T is a partial subsemigroup of S and the inclusion map is a filtered semigroup homomorphism. In other words, the distinguished filter \mathfrak{G} on T is just the trace $\mathfrak{F}|_T := \{A \cap T : A \in \mathfrak{F}\}$ of \mathfrak{F} on T . This allows one to identify $\beta_{\mathfrak{G}}T$ with a closed subsemigroup of $\beta_{\mathfrak{F}}S$. In this situation we denote $\beta_{\mathfrak{F}|_T}T$ by $\beta_{\mathfrak{F}}T$.

Suppose that $(S, +, \mathfrak{F})$ is a filtered semigroup. We denote by $\mathcal{S}(S, +, \mathfrak{F})$ the set of filtered subsemigroups of S . The set $\mathcal{S}(S, +, \mathfrak{F})$ is endowed with a canonical ordering obtained by setting $S_0 \leq S_1$ if and only if, for every $s_0 \in S_0$ and $s_1 \in S_1$, $s_0 + s_1$ and $s_1 + s_0$ belong to S_0 whenever they are defined. The map $S_0 \mapsto \beta_{\mathfrak{F}}S_0$ defines an order-preserving function from the space $\mathcal{S}(S, +, \mathfrak{F})$ of filtered subsemigroups of S to the space $\mathcal{S}(\beta_{\mathfrak{F}}S)$ of closed subsemigroups of $\beta_{\mathfrak{F}}S$.

Example 5.3. Suppose that $(S, +)$ is a semigroup and (x_n) is a sequence in S . Consider then the filter \mathfrak{F} on S generated by sets of the form $\text{FS}(x_n)_{n \geq r} = \{x_{n_0} + x_{n_1} + \dots + x_{n_\ell} : \ell \in \omega, r \leq n_0 < \dots < n_\ell \in \omega\}$ for $r \in \omega$. Then $(S, +, \mathfrak{F})$ is a filtered semigroup. In this case $\beta_{\mathfrak{F}}S$ is the closed subsemigroup of βS consisting of ultrafilters on S that contain $\text{FS}(x_n)_{n \geq r}$ for every $r \in \omega$.

Example 5.4. Suppose that S is a semigroup, and \mathfrak{F} is the filter of cofinite sets. Then $\beta_{\mathfrak{F}}S$ is the set of nonprincipal ultrafilters over S . If S is either left or right cancellative, then $(S, \mathfrak{F}, +)$ is a filtered semigroup; see also [16, Corollary 4.29].

Remark 5.5. Any filtered semigroup S can be seen as a filtered semigroup *where moreover the operation is everywhere defined* by adding an extra *absorbing element*. Let $(S, +, \mathfrak{F})$ be a filtered partial semigroup such that S does not contain the symbol 0. Then one can consider the semigroup S_0 by setting $0 + x = x + 0 = 0$ for any $x \in S_0$ and $x + y = 0$ whenever $x, y \in S$ and $x + y$ is not defined in S . Then one can consider the filter \mathfrak{F}_0 of subsets of S_0 generated by \mathfrak{F} . This gives a filtered semigroup $(S_0, +, \mathfrak{F}_0)$ where moreover the operation on S_0 is everywhere defined. The compact right topological semigroup $\beta_{\mathfrak{F}_0}S_0$ is canonically isomorphic to $\beta_{\mathfrak{F}}S$.

If S, T are filtered semigroups and $\sigma : S \rightarrow T$ is a filtered semigroup homomorphism, then one can canonically extend σ to a filtered semigroup homomorphism $\sigma_0 : S_0 \rightarrow T_0$ such that $\sigma(0) = 0$.

5.2. Actions of ordered sets on filtered semigroups. Suppose that \mathbb{P} is an ordered set, and $(S, +, \mathfrak{F})$ is a filtered semigroup. We denote by $\text{End}(S, +, \mathfrak{F})$ the space of filtered semigroup homomorphisms $\tau : S \rightarrow S$. Observe that $\text{End}(S, +, \mathfrak{F})$ is a semigroup with respect to composition.

Definition 5.6. An *action* α of \mathbb{P} on $(S, +, \mathfrak{F})$ is given by

- an order-preserving function $\mathbb{P} \rightarrow \mathcal{S}(S, +, \mathfrak{F})$, $t \mapsto S_t$, and
- a subsemigroup $\mathcal{F}_\alpha \subset \text{End}(S, +, \mathfrak{F})$,

such that for every $\tau \in \mathcal{F}_\alpha$ there exists a function $f_\tau : \mathbb{P} \rightarrow \mathbb{P}$ —which we call the *spine* of τ —such that τ maps S_t to $S_{f_\tau(t)}$ for every $t \in \mathbb{P}$, and such that $\tau(s) = s$ for any $s \in S_t$ and $t \in T$ such that $f_\tau(t) = t$.

As in the case of actions on adequate partial semigroups, an action α of \mathbb{P} on $(S, +, \mathfrak{F})$ induces an action—in the sense of Definition 2.2—of \mathbb{P} on the compact right topological semigroup $X = \beta_{\mathfrak{F}} S$, which we still denote by α . Such an action is obtained by setting $X_t := \beta_{\mathfrak{F}} S_t$ and $\tau : \beta_{\mathfrak{F}} S \rightarrow \beta_{\mathfrak{F}} S$ to be the unique extension of τ to a continuous partial semigroup endomorphism of $\beta_{\mathfrak{F}} S$ for every $\tau \in \mathcal{F}_\alpha$. Consistently with the notation used in Subsection 2.2, we denote by $(\beta_{\mathfrak{F}} S)_\alpha$ the set of functions $\xi : \mathbb{P} \rightarrow \beta_{\mathfrak{F}} S$ such that $\xi(t) \in \beta_{\mathfrak{F}} S_t$ and $\tau \circ \xi = \xi \circ f_\tau$ for every $\tau \in \mathcal{F}_\alpha$. An order-preserving idempotent in $(\beta_{\mathfrak{F}} S)_\alpha$ is an element ξ of $(\beta_{\mathfrak{F}} S)_\alpha$ such that $\xi(t)$ is an idempotent element of $\beta_{\mathfrak{F}} S_t$ and $\xi(t) \leq \xi(t_0)$ whenever $t, t_0 \in \mathbb{P}$ are such that $t \leq t_0$.

The same proof as Theorem 3.2 gives the following.

Theorem 5.7. Suppose that $k, m \in \mathbb{N}$ and $\lambda : m \rightarrow k$ is a function. For $i \in k$ let $(S_i, +_i, \mathfrak{F}_i)$ be a filtered semigroup, \mathbb{P}_i a finite ordered set, and α_i an action of \mathbb{P}_i on $(S_i, +_i, \mathfrak{F}_i)$. Suppose that $\xi_i \in (\gamma S_i)_{\alpha_i}$ is an order-preserving idempotent for $i \in k$. Set $S := S_{\lambda(0)} \times \cdots \times S_{\lambda(m-1)}$ and suppose that c is a finite coloring of S . Extend c canonically to a coloring of βS . Fix a sequence $(\psi_{i,n}^{(\mathfrak{F})})$ of functions $\psi_{i,n} : (S_i^{\mathbb{P}_i})^n \rightarrow \mathfrak{F}_i$, and sequence $(\psi_{i,n}^{(\mathcal{F})})$ of functions $\psi_{i,n} : (S_i^{\mathbb{P}_i})^n \rightarrow [\mathfrak{F}_i]^{<\aleph_0}$. Then there exist sequences $(x_{i,n})_{n \in \omega}$ of functions $x_{i,n} : \mathbb{P}_i \rightarrow S_i$ such that

- $x_{i,n}(t) \in S_{i,t} \cap \psi_{i,n}^{(\mathfrak{F})}(x_{i,0}, \dots, x_{i,n-1})$ for every $n \in \omega$, $i \in k$, and $t \in \mathbb{P}_i$; and
- for any $\ell \in \omega$, $i \in k$, finite subsets $F_0 < F_1 < \cdots < F_{m-1}$ of ω , sequences $(t_{i,n})_{n \in \omega}$ in \mathbb{P}_i and $(\tau_{i,n})_{n \in \omega}$ in \mathcal{F}_{α_i} such that $\tau_{i,n} \in \psi_{i,n}^{(\mathcal{F})}(x_{i,0}, \dots, x_{i,n-1})$ for every $n \in \omega$, if $\{f_{\tau_{\lambda(s),d}}(t_{\lambda(s),d}) : d \in F_s\}$ is a chain in $\mathbb{P}_{\lambda(s)}$ with least element t_s for every $s \in m$, then the color of

$$\left(\sum_{d \in F_s}^{\lambda(s)} \tau_{\lambda(s),d} (x_{\lambda(s),d} (t_{\lambda(s),d})) \right)_{s \in m}$$

is the same as the color of $\xi_{\lambda(0)}(t_0) \otimes \xi_{\lambda(1)}(t_1) \otimes \cdots \otimes \xi_{\lambda(m-1)}(t_{m-1})$.

Suppose now that T is a finite rooted tree endowed with the canonical tree ordering.

Definition 5.8. Suppose that α is an action of T on a filtered semigroup $(S, +, \mathfrak{F})$ given by a semigroup $\mathcal{F}_\alpha \subset \text{End}(S, +, \mathfrak{F})$. We say that α is *Ramsey* if, for any $\tau \in \mathcal{F}_\alpha$, the corresponding spine $f_\tau : T \rightarrow T$ is a regressive homomorphism, and for any $C \in \mathfrak{F}$, for any finite coloring c of S , and for any finite subset \mathcal{F}_0 of \mathcal{F}_α , there exists a function $x : T \rightarrow S$ such that, for any $\tau \in \mathcal{F}_0$ and $t \in T$, $x(t) \in S_t \cap C$ and the color of $\tau(x(t))$ depends only on $f_\tau(t)$.

The same proofs as Theorem 3.4 and Theorem 5.7 give the following.

Theorem 5.9. Suppose that α is an action of a finite rooted tree on a filtered semigroup $(S, +, \mathfrak{F})$ such that, for every $\tau \in \mathcal{F}_\alpha$, the corresponding spine f_τ is a regressive homomorphism. The following statements are equivalent:

- (1) α is Ramsey;
- (2) the action of T on $\beta_{\mathfrak{F}} S$ induced by α is Ramsey;
- (3) for any finite coloring c of S , sequence (ψ_n) of functions $\psi_n^{(\mathfrak{F})} : (S^T)^n \rightarrow \mathfrak{F}$, and sequence $(\psi_n^{(\mathcal{F})})$ of functions $\psi_n^{(\mathcal{F})} : (S^T)^n \rightarrow [\mathcal{F}]^{<\aleph_0}$, there exist functions $x_n : T \rightarrow S$ such that
 - $x_n(t) \in S_t \cap \psi_n^{(\mathfrak{F})}(x_0, \dots, x_{n-1})$ for every $n \in \omega$ and $t \in T$; and
 - for any $\ell \in \omega$, $n_0 < n_1 < \cdots < n_\ell \in \omega$, $t_i \in T$ for $i \leq \ell$, and $\tau_i \in \mathcal{F}_{\alpha_i}$ for $i \leq \ell$, if $\{f_{\tau_i}(t_i) : i \leq \ell\}$ is a chain in T with least element t , then the color of $\tau_0(x_{n_0}(t_0)) + \cdots + \tau_\ell(x_{n_\ell}(t_\ell))$ depends only on t .

Theorem 5.10. *Suppose that $k, m \in \mathbb{N}$ and $\lambda : m \rightarrow k$ is a function. For $i \in k$ let $(S_i, +_i, \mathfrak{F}_i)$ be a filtered semigroup, T_i a finite rooted tree set, and α_i a Ramsey action of T_i on $(S_i, +_i, \mathfrak{F}_i)$. Set $S := S_{\lambda(0)} \times \cdots \times S_{\lambda(m-1)}$ and suppose that c is a finite coloring of S . Fix a sequence $(\psi_{i,n}^{(\mathfrak{F})})$ of functions $\psi_{i,n}^{(\mathfrak{F})} : (S_i^{T_i})^n \rightarrow \mathfrak{F}_i$, and sequence $(\psi_{i,n}^{(\mathcal{F})})$ of functions $\psi_{i,n}^{(\mathcal{F})} : (S_i^{T_i})^n \rightarrow [\mathcal{F}_i]^{<\aleph_0}$. Then there exist sequences $(x_{i,n})_{n \in \omega}$ of functions $x_{i,n} : T_i \rightarrow S_i$ such that*

- $x_{i,n}(t) \in S_{i,t} \cap \psi_{i,n}^{(\mathfrak{F})}(x_{i,0}, \dots, x_{i,n-1})$ for every $n \in \omega$, $i \in k$, and $t \in T_i$; and
- for any $\ell \in \omega$, $i \in k$, $F_0 < F_1 < \cdots < F_{m-1}$, sequences $(t_{i,n})_{n \in \omega}$ in T_i and $(\tau_{i,n})_{n \in \omega}$ in \mathcal{F}_{α_i} such that $\tau_{i,n} \in \psi_{i,n}^{(\mathcal{F})}(x_0, \dots, x_{n-1})$ for every $n \in \omega$, if $\{f_{\tau_{\lambda(s),d}}(t_{\lambda(s),d}) : d \in F_s\}$ is a chain in $T_{\lambda(s)}$ with least element t_s for every $s \in m$, then the color of

$$\left(\sum_{d \in F_s}^{\lambda(s)} \tau_{\lambda(s),d}(x_{\lambda(s),d}(t_{\lambda(s),d})) \right)_{s \in m}$$

depends only on $(t_s)_{s \in m}$.

5.3. The Hales-Jewett theorem for nonlocated words. Suppose that L is a set (alphabet). Let W_L be the set of finite strings of elements of L (words). Let also v be a symbol not in L (variable), and W_{Lv} be the set of finite strings of elements of L where v appears (variable words). Then $S := W_L \cup W_{Lv}$ is a cancellative semigroup with respect to the operation $+$ of concatenation. Thus $(S, \mathfrak{F}, +)$ is a filtered semigroup, where \mathfrak{F} is the filter of cofinite subsets of S .

Any element $a \in L$ defines a variable substitution map $\tau_a : S \rightarrow S$ mapping any word w to the word obtained from w by replacing every occurrence of the variable v (if any) by a . This defines a Ramsey action of the rooted tree with two nodes on $(S, \mathfrak{F}, +)$. It follows from Proposition 2.7 that such an action is Ramsey. Therefore Theorem 5.7 in this case recovers the infinite Hales-Jewett theorem (for nonlocated words) [4, Theorem 1.1]—see also [24, Theorem 2.35].

One can also consider a generalization of such a result for sets of variables indexed by a tree similar to Theorem 4.5. Suppose that T is a finite rooted tree with root r , and let T^+ be the set of nodes of T different from the root. We think of the elements of T^+ as variables. Let also L be a set. Define $W_{L,r}^T$ to be the set of words with symbols from L . For $t \in T^+$ let $W_{L,t}^T$ be the set of words b with symbols from $L \cup T^+$ such that the variables that appear in b form a nonempty chain in T with least element t . We then let W_L^T be the set of all words in $L \cup T^+$. This is a cancellative semigroup with respect to the concatenation operation $+$. We regard $(W_L^T, +)$ as a filtered semigroup endowed with the filter \mathfrak{F} of cofinite subsets of W_L^T .

Definition 5.11. A variable substitution map is a function σ from $T^+ \cup L$ to $T^+ \cup L$ such that

- $\sigma|_L$ is the identity function of L , and
- the function $f_\sigma : T \rightarrow T$ defined by

$$t \mapsto \begin{cases} \sigma(t) & \text{if } t \in T^+ \text{ and } \sigma(t) \in T^+, \\ r & \text{otherwise} \end{cases}$$

is a regressive homomorphism of T .

Any variable substitution map σ defines a filtered semigroup homomorphism τ_σ of $(W_L^T, +, \mathfrak{F})$ defined as follows. For every $b \in W_L^T$, $\tau_\sigma(b)$ is the word obtained from b replacing every occurrence of x with $\sigma(x)$ for every $x \in T^+ \cup L$. The same argument as in Subsection 4.2 shows that such an action is Ramsey; see Definition 5.8.

Therefore applying Theorem 5.10 one obtains the following result, which is a generalization of the infinite Hales-Jewett theorem for nonlocated words [4, Theorem 1.1].

Theorem 5.12. *Suppose that T is a finite tree, L is a set, and c is a finite coloring of W_L^T . For any sequence (d_n) in ω and sequence (\mathcal{F}_n) where \mathcal{F}_n is a finite set of variable substitution maps for T, L as in Definition 5.11, there exist sequences $(b_{t,n})_{n \in \omega}$ in $W_{L,t}^T$ for $t \in T$ such that*

- for every $n \in \omega$ and $t \in T$, the length of $b_{t,n}$ is at least d_n , and
- for every $\ell \in \omega$, $n_0 < \cdots < n_\ell \in \omega$, $t_0, \dots, t_\ell \in T$, variable substitution maps $\sigma_j \in \mathcal{F}_j$ for $j \leq \ell$ such that $\{f_{\sigma_0}(t_0), \dots, f_{\sigma_\ell}(t_\ell)\}$ is a chain in T with least element $t \in T$ one has that the color of

$$\tau_{\sigma_0}(b_{t_0,n_0}) + \cdots + \tau_{\sigma_\ell}(b_{t_\ell,n_\ell})$$

depends only on t .

The infinite Hales–Jewett theorem for nonlocated words [4, Theorem 1.1] is the particular instance of Theorem 4.5 where T is the rooted tree with only two nodes.

More generally from Corollary 3.7 one can deduce the following generalization of Theorem 4.5.

Theorem 5.13. *Suppose that $m, k \in \mathbb{N}$ and $\lambda : m \rightarrow k$ is a function. Let T_i be a finite rooted tree and L_i be a set for $i \in k$. Fix a finite coloring c of $W_{L_{\lambda(0)}}^{T_{\lambda(0)}} \times \cdots \times W_{L_{\lambda(m-1)}}^{T_{\lambda(m-1)}}$. For any sequence (d_n) of elements of ω , $i \in k$, sequence $(\mathcal{F}_{i,n})$ of finite sets of variable substitution maps for T_i, L_i in the sense of Definition 5.11, and for every $t \in T_i$, there exist sequences $(b_{i,t,n})_{n \in \omega}$ in $\text{FIN}_{L_i,t}^{T_i}$ such that*

- the length of $b_{i,t,n}$ is at least d_n ,
- for any $F_0 < F_1 < \cdots < F_{m-1}$, nodes $t_{i,d}$ of T_i and variable substitution maps $\sigma_{i,d} \in \mathcal{F}_{i,d}$ for $d \in \omega$ and $i \in k$ such that $\{f_{\sigma_{\lambda(s),d}}(t_{\lambda(s),d}) : d \in F_s\}$ is a chain in $T_{\lambda(s)}$ with least element t_s for every $s \in m$, the color of

$$\left(\sum_{d \in F_s} \tau_{\sigma_{\lambda(s),d}}(b_{f(s),t_{\lambda(s),d}}) \right)_{s \in m}$$

depends only on $(t_s)_{s \in m}$.

6. A POLYNOMIAL GOWERS' RAMSEY THEOREM

6.1. Extended polynomials. We follow the approach and notation of [7]. Suppose that (S, \mathfrak{F}) is a filtered set. Let \mathfrak{S} be a collection of partial semigroups operations on S such that, for every $+$ in \mathfrak{S} , $(S, +, \mathfrak{F})$ is a filtered semigroup. The space $\beta_{\mathfrak{F}}S$ of cofinite ultrafilters over (S, \mathfrak{F}) is then a semigroup with respect to the canonical extension to βS of any of the operations in \mathfrak{S} . In the following if $a \in S$ and \mathfrak{G} is a filter over S we let $a + \mathfrak{G}$ be the image of \mathfrak{G} under the left translation map $x \mapsto a + x$. Explicitly, $C \in a + \mathfrak{G}$ if and only if $-a + C = \{x \in S : a + x \in C\} \in \mathfrak{G}$. Similarly we define $\mathfrak{G} + a$ in terms of the right translation map $x \mapsto x + a$.

One can define as in [7, Definition 3.1] the set \mathfrak{P} of *extended polynomials* in the variables x_0, x_1, \dots corresponding to the set \mathfrak{S} by induction on the degree as follows:

- (1) x_n is an extended polynomial for every $n \in \omega$;
- (2) if $+$ in \mathfrak{S} and $p(x_0, \dots, x_{n-1})$ and $q(x_n, \dots, x_{n+m-1})$ are extended polynomials, then $p(x_0, \dots, x_{n-1}) + q(x_n, \dots, x_{n+m-1})$ is an extended polynomial;
- (3) if $+$ in \mathfrak{S} , $p(x_0, \dots, x_{n-1})$ is an extended polynomial, and $a \in \mathfrak{S}$ is such that $\mathfrak{F} + a \supset \mathfrak{F}$, then $p(x_0, \dots, x_{n-1}) + a$ is an extended polynomial;
- (4) if $+$ in \mathfrak{S} , $p(x_0, \dots, x_{n-1})$ is an extended polynomial, and $a \in \mathfrak{S}$ is such that $a + \mathfrak{F} \supset \mathfrak{F}$ then $a + p(x_0, \dots, x_{n-1})$ is an extended polynomial.

Any extended polynomial $p(x_0, \dots, x_{n-1})$ defines a polynomial mapping $f_p : S^n \rightarrow S$ in the obvious way. One can then consider its canonical extension to a continuous function $f_p : \beta(S^n) \rightarrow \beta S$. One can also evaluate a polynomial $p(x_0, \dots, x_{n-1})$ at a tuple $(\mathcal{U}_0, \dots, \mathcal{U}_{n-1})$ of elements of $\beta_{\mathfrak{F}}S$ by interpreting the operations in \mathfrak{S} as their canonical extensions to right topological semigroup operations on $\beta_{\mathfrak{F}}S$. The following proposition is the analog of Theorem [7, Theorem 3.2] in this context. The proof is entirely analogous, and it is presented here for convenience of the reader.

Proposition 6.1. *Suppose that $p(x_0, \dots, x_{n-1})$ is an extended polynomial, and $f_p : S^n \rightarrow S$ is the corresponding polynomial mapping. Then for every $\mathcal{U}_0, \dots, \mathcal{U}_{n-1} \in \beta_{\mathfrak{F}}S$ we have that $p(\mathcal{U}_0, \dots, \mathcal{U}_{n-1}) \in \beta_{\mathfrak{F}}S$ is equal to $f_p(\mathcal{U}_0 \otimes \cdots \otimes \mathcal{U}_{n-1})$.*

Proof. In the course of the proof, we denote tuples of variables x_0, \dots, x_{n-1} and y_0, \dots, y_{m-1} by \bar{x} and \bar{y} respectively. The proof is naturally by induction on the complexity of the given extended polynomial. Clearly the conclusion is true for the polynomial x_n .

Suppose that the conclusion is true for $p(\bar{x})$ and $q(\bar{y})$. Fix $+$ in \mathfrak{S} and set $r(\bar{x}, \bar{y}) := p(\bar{x}) + q(\bar{y})$. Let $\mathcal{U}_0, \dots, \mathcal{U}_{n-1}$ and $\mathcal{V}_0, \dots, \mathcal{V}_{m-1}$ be elements of $\beta_{\mathfrak{F}}(S)$. Set $\mathcal{U} := \mathcal{U}_0 \otimes \cdots \otimes \mathcal{U}_{n-1}$ and $\mathcal{V} := \mathcal{V}_0 \otimes \cdots \otimes \mathcal{V}_{m-1}$. Then we have that

$$r(\mathcal{U}_0, \dots, \mathcal{U}_{n-1}, \mathcal{V}_0, \dots, \mathcal{V}_{m-1}) = p(\mathcal{U}_0, \dots, \mathcal{U}_{n-1}) + q(\mathcal{V}_0, \dots, \mathcal{V}_{m-1}) \in \beta_{\mathfrak{F}}S$$

since $(\beta_{\mathfrak{F}}S, +)$ is a semigroup. We now want to prove that $r(\mathcal{U}_0, \dots, \mathcal{U}_{n-1}, \mathcal{V}_0, \dots, \mathcal{V}_{m-1}) = f_r(\mathcal{U} \otimes \mathcal{V})$. We denote by \bar{a} and \bar{b} tuples (a_0, \dots, a_{n-1}) and (b_0, \dots, b_{m-1}) of elements of S . Suppose that $C \in S$. We have $C \in f_r(\mathcal{U} \otimes \mathcal{V})$ if and only if $(\mathcal{U} \otimes \mathcal{V})(\bar{a}, \bar{b})$, $f_r(\bar{a}, \bar{b}) \in C$, if and only if $(\mathcal{U}\bar{a}), (\mathcal{V}\bar{b})$, $f_r(\bar{a}, \bar{b}) \in C$, if and only if $(\mathcal{U}\bar{a}), (\mathcal{V}\bar{b})$, $f_p(\bar{a}) + f_q(\bar{b}) \in C$, if and only if $(f_p(\mathcal{U})x), (f_q(\mathcal{V})y), x + y \in C$, if and only if (using the inductive assumption) $(p(\mathcal{U}_0, \dots, \mathcal{U}_{n-1})x), (q(\mathcal{V}_0, \dots, \mathcal{V}_{m-1})y), x + y \in C$, if and only if $((p(\mathcal{U}_0, \dots, \mathcal{U}_{n-1}) + q(\mathcal{V}_0, \dots, \mathcal{V}_{m-1}))z)$

$z \in C$, if and only if $C \in r(\mathcal{U}_0, \dots, \mathcal{U}_{n-1}, \mathcal{V}_0, \dots, \mathcal{V}_{m-1}) = p(\mathcal{U}_0, \dots, \mathcal{U}_{n-1}) + q(\mathcal{V}_0, \dots, \mathcal{V}_{m-1})$. This concludes the proof that $f_r(\mathcal{U} \otimes \mathcal{V}) = r(\mathcal{U}_0, \dots, \mathcal{U}_{n-1}, \mathcal{V}_0, \dots, \mathcal{V}_{m-1})$.

Suppose now that the conclusion is true for $p(\bar{y})$ and $a \in S$. Let $+ \in \mathfrak{S}$ be such that $a + \mathfrak{F} \supset \mathfrak{F}$. We prove that the conclusion holds for $q(\bar{y}) := a + p(\bar{y})$. The proof that the conclusion holds for $p(\bar{y}) + a$ under the assumption that $\mathfrak{F} + a \supset \mathfrak{F}$ is analogous. Suppose that $\mathcal{V}_0, \dots, \mathcal{V}_{m-1} \in \beta_{\mathfrak{F}} S$. Then by recursive assumption we have that $p(\mathcal{V}_0, \dots, \mathcal{V}_{m-1}) \in \beta_{\mathfrak{F}} S$. Therefore $a + p(\mathcal{V}_0, \dots, \mathcal{V}_{m-1}) \supset a + \mathfrak{F} \supset \mathfrak{F}$ and hence $a + p(\mathcal{V}_0, \dots, \mathcal{V}_{m-1}) \in \beta_{\mathfrak{F}} S$. Set $\mathcal{V} := \mathcal{V}_0 \otimes \dots \otimes \mathcal{V}_{m-1}$. We now prove that $f_q(\mathcal{V}) = a + p(\mathcal{V}_0, \dots, \mathcal{V}_{m-1})$. Suppose that C is a subset of S . We have that $C \in f_q(\mathcal{V})$ if and only if $(\mathcal{V}\bar{b}) f_q(\bar{b}) \in C$ if and only if $(\mathcal{V}\bar{b}) a + f_p(\bar{b}) \in C$ if and only if $(f_p(\mathcal{V})z) a + z \in C$ if and only if (using the inductive hypothesis) $(p(\mathcal{V}_0, \dots, \mathcal{V}_{n-1})z) a + z \in C$, if and only if $((a + p(\mathcal{V}_0, \dots, \mathcal{V}_{n-1}))z) z \in C$. This shows that $f_q(\mathcal{V}) = a + p(\mathcal{V}_0, \dots, \mathcal{V}_{n-1}) = q(\mathcal{V}_0, \dots, \mathcal{V}_{n-1})$, concluding the proof. \square

6.2. A polynomial Ramsey theorem for actions of trees. In this section we provide a generalization of the polynomial Milliken-Taylor theorem of Bergelson, Hindman, and Williams [7, Corollary 3.5] to the setting of actions of trees on filtered semigroups. In the following we suppose as above that $m, k \in \mathbb{N}$ and $\lambda : m \rightarrow k$ is a function. Suppose that (S, \mathfrak{F}) is a filtered set, and \mathfrak{S} is a collection of semigroup operations as in Subsection 6.1. Let $p(x_0, \dots, x_{m-1})$ be an extended polynomial defined with respect to \mathfrak{S} and \mathfrak{F} . For $i \in k$ we let T_i be a finite tree and $+_i$ be an element of \mathfrak{S} . Consider for $i \in k$ a Ramsey action α_i of T_i on $(S, +_i, \mathfrak{F})$ in the sense of Definition 5.6. This is given by an order-preserving assignment $T \rightarrow \mathcal{S}(S, +_i, \mathfrak{F})$, $t \mapsto S_{i,t}$, and a semigroup \mathcal{F}_{α_i} of filtered semigroup homomorphisms of $(S, +_i, \mathfrak{F})$ such that every $\tau \in \mathcal{F}_{\alpha_i}$ maps $S_{i,t}$ to $S_{i,f_\tau(t)}$, where $f_\tau : T_i \rightarrow T_i$ is a regressive homomorphism. We will furthermore assume that \mathcal{F}_{α_i} is finite for every $i \in k$.

Let S_{α_i} be the set of functions $x : T_i \rightarrow S$ such that $x(t) \in S_{i,t}$ for every $t \in T_i$. Given a subset A of ω , a sequence $(x_n)_{n \in A}$ in S_{α_i} , and $t \in T_i$, we define the $(\alpha_i, +_i, t)$ -semigroup $S_{\alpha_i, +_i, t}(x_n)_{n \in A}$ generated by $(x_n)_{n \in A}$ to be the collection of elements of $S_{i,t}$ of the form

$$\sum_{d \in F}^i \tau_d(x_d(t_d))$$

for finite $F \subset A$, and families $(t_d)_{d \in A}$ in T_i , $(\tau_d)_{d \in A}$ in \mathcal{F}_{α_i} such that $\{f_{\tau_d}(t_d) : d \in F\}$ is a chain in T_i with least element t . A sequence (y_n) in S_{α_i} is an $(\alpha_i, +_i)$ -subsystem of (x_n) if $\text{dom}(y_n) < \text{dom}(y_{n+1})$ and $y_n(t) \in S_{\alpha_i, +_i, t}(x_\ell)_{\ell \in \omega}$ for every $n \in \omega$.

Theorem 6.2. *Fix, for every $i \in k$, a sequence $(x_{i,n})_{n \in \omega}$ in S_{α_i} . Let $A_{(t_s)_{s \in m}}$ be a subset of S whenever $t_s \in T_{\lambda(s)}$ for $s \in m$. The following assertions are equivalent:*

- (1) *there exist order-preserving idempotent elements $\xi_i \in (\beta_{\mathfrak{F}_i} S_i)_{\alpha_i}$ such that $S_{\alpha_i, +_i, t}(x_{i,n})_{n \geq r} \in \xi_i(t)$ for every $i \in k$, $t \in T_i$, $r \in \omega$, and $A_{(t_s)_{s \in m}} \in p(\xi_{\lambda(0)}(t_0), \dots, \xi_{\lambda(m-1)}(t_{m-1}))$ whenever $t_s \in T_{\lambda(s)}$ for $s \in m$;*
- (2) *for any finite coloring c of S , for each $i \in k$, sequence $(\psi_{i,n}^{(\mathfrak{F})})$ of functions $\psi_{i,n}^{(\mathfrak{F})} : (S^{T_i})^n \rightarrow \mathfrak{F}_i$, there exists an $(\alpha_i, +_i)$ -tetris subsystem $(y_{i,n})$ of $(x_{i,n})$ such that for every choice of $t_i \in T_i$ for $i \in k$ one has that*
 - $y_{i,n}(t) \in S_{i,t} \cap \psi_{i,n}^{(\mathfrak{F})}(x_{i,0}, \dots, x_{i,n-1})$ for every $t \in T_i$;
 - $p\left(\sum_{d \in F_s}^{\lambda(s)} \tau_{\lambda(s),d}(y_{\lambda(s),d}(t_{\lambda(s),d}))\right)_{s \in m} \in A_{(t_s)_{s \in m}}$ whenever $F_0 < F_1 < \dots < F_{m-1}$ are finite subsets of ω , $(t_{i,d})_{d \in \omega}$ is a sequence in T_i and $(\tau_{i,d})_{d \in \omega}$ is a sequence in \mathcal{F}_{α_i} for $i \in k$ such that $\{f_{\tau_{\lambda(s),d}}(t_{\lambda(s),d}) : d \in F_s\}$ is a chain in $T_{\lambda(s)}$ with least element t_s ;
 - the color of $\tau(y_{i,n}(t))$ depends only on $f_\tau(t)$ for every $i \in k$ and $n \in \omega$.
- (3) *The same as (2) where moreover $S_{\alpha_i, +_i, t}(y_{i,n})_{n \in \omega}$ is monochromatic for every $i \in k$, $t \in T_i$.*

Proof. The proof of the implication (1) \Rightarrow (3) is the same as the proof of Theorem 3.2. The implication (3) \Rightarrow (2) is obvious. We consider the implication (2) \Rightarrow (1). Define for $i \in k$ and $t \in T_i$ the filter $\mathfrak{G}_{i,t}$ on S consisting of the sets of the form $S_{\alpha_i, +_i, t}(y_{i,n})_{n \geq r} \cap C$ for $r \in \omega$ and $C \in \mathfrak{F}$. Observe that such sets are nonempty by assumption. Define then $\beta_{\mathfrak{G}_{i,t}} S$ to be the set of ultrafilters on S that contain $\mathfrak{G}_{i,t}$. The Ramsey action α_i of T_i on S induces a Ramsey action α_i of T_i on βS given by the map $t \mapsto \beta_{\mathfrak{G}_{i,t}} S_{i,t}$ and the collection of semigroup homomorphisms $\tau : \beta S \rightarrow \beta S$ given by the continuous extensions of elements τ of \mathcal{F}_{α_i} . The assumption implies by compactness that the set of functions $\xi_i : T_i \rightarrow \beta S$ such that $\xi_i(t) \in \beta_{\mathfrak{G}_{i,t}} S_{i,t}$ and $\tau \circ \xi_i = \xi_i \circ f_\tau$ for every $\tau \in \mathcal{F}_{\alpha_i}$. Hence by Proposition 2.5 one can find such a function $\xi_i : T_i \rightarrow \beta S$ such that moreover $\xi_i(t)$ is an idempotent element of $\beta_{\mathfrak{G}_{i,t}} S_{i,t}$ and $\xi_i(t) \leq \xi_i(t_0)$ whenever $t, t_0 \in T$ are such that $t \leq t_0$. Finally observe that since $\xi_i(t) \in \beta_{\mathfrak{G}_{i,t}} S_{i,t}$ we have that $\text{TS}_{\alpha_i, +_i, t}(y_{i,n})_{n \geq r} \in \xi_i(t)$ for every $r \in \omega$. Since $(y_{i,n})$ is an $(\alpha_i, +_i)$ -tetris subsystem of $(x_{i,n})$, this implies that $\text{TS}_{\alpha_i, +_i, t}(x_{i,n})_{n \geq r} \in \xi_i(t)$. This concludes the proof. \square

The polynomial Milliken–Taylor theorem of Bergelson–Hindman–Williams [7, Corollary 3.5] is the particular instance of Theorem 6.2 where each tree T_i has only one node, the filter \mathfrak{F} on S is the trivial filter $\{S\}$, each semigroup operation $+$ $\in \mathfrak{S}$ on S is everywhere defined.

7. APPLICATIONS TO COMBINATORIAL CONFIGURATIONS IN DELTA SETS

7.1. Configurations in delta sets in amenable groups. Suppose that (G, \cdot) is a discrete *amenable* group. Recall that G is endowed with a canonical notion of *density*—also known as Banach density—defined as follows. Let A be a subset of G , and α be a positive real number. Then the density $d(A)$ of A is larger than or equal to α if and only if for any finite subset F of G and any $\varepsilon > 0$ there exists a finite subset I of G such that $|A \cap I| \geq (\alpha - \varepsilon)|I|$ and $|gI \triangle I| < \varepsilon|I|$ for every $g \in F$. A subset A of G has positive density if $d(A) > 0$. We recall for future use the following *correspondence principle*, which is established in [3, Lemma 4.6].

Lemma 7.1. Suppose that $A \subset G$ has positive density. Then there exist a standard Borel probability space (X, μ) , a measure-preserving action $g \mapsto \tilde{g} \in \text{Aut}(X, \mu)$ of G on (X, μ) , and a Borel subset \tilde{A} of X such that $\mu(\tilde{A}) = d(A)$, and for any finite subset F of G , one has that

$$d\left(\bigcap_{g \in F} gA\right) \geq \mu\left(\bigcap_{g \in F} \tilde{g}\tilde{A}\right).$$

The following result about sets of positive density was proved by Furstenberg as an application of the corresponding principle and Hindman’s theorem on finite unions [11]. Recall that the delta set AA^{-1} associated with a subset A of G is the set $\{gh^{-1} : g, h \in A\} \subset G$.

Theorem 7.2 (Furstenberg). *Suppose that $A \subset G$ has positive density, and (g_n) is a sequence of elements of G . Then there exists a sequence (F_n) of finite nonempty subsets of ω such that $F_n < F_{n+1}$ for every $n \in \omega$ and such that, for every finite subset E of ω one has that*

$$\prod_{d \in E} \prod_{i \in F_d} g_i \in AA^{-1}.$$

In the statement of Theorem 7.2 and in the following, we convene that, for a given finite nonempty subset F of ω and family $(g_i)_{i \in F}$ in G , the product $\prod_{i \in F} g_i$ denotes the element $g_{i_0} \cdots g_{i_{k-1}}$ of G , where $(i_s)_{s \in k}$ is the increasing enumeration of F . If F is empty, then $\prod_{i \in F} g_i$ denotes the identity of G .

Fix now a finite rooted tree T . Let T^+ be the set of nodes in T different from the root. Recall that FIN^T denotes the adequate partial semigroup of functions $b : \text{dom}(b) \rightarrow T^+$, where $\text{dom}(b)$ is a (possibly empty) finite subset of ω . If $f : T \rightarrow T$ is a regressive homomorphism—see Definition 2.3—then $\tau_f : \text{FIN}^T \rightarrow \text{FIN}^T$ is the corresponding adequate partial semigroup homomorphism defined by $\tau_f(b) : \{n \in \text{dom}(b) : f(b(n)) \in T^+\} \rightarrow T^+, n \mapsto f(b(n))$. For $t \in T^+$ we let FIN_t^T be the set of $b \in \text{FIN}^T$ such that the range of b is a nonempty chain in T with least element t . If r is the root of T , then FIN_r^T contains only the empty function.

Theorem 7.3. *Suppose that $A \subset G$ has positive density. Let (g_n) be a sequence of functions $g : T^+ \rightarrow G$. There exists a sequence of functions $b_n : T \rightarrow \text{FIN}^T$, $t \mapsto b_{n,t} \in \text{FIN}_t^T$ such that $\text{dom}(b_{n,t}) < \text{dom}(b_{n+1,t})$ and for any finite nonempty subset F of ω , regressive homomorphisms $f_d : T \rightarrow T$ and nodes $t_d \in T$ for $d \in F$ such that $\{f_d(t_d) : d \in F\}$ is a chain in T , one has that*

$$\prod_{d \in F} \prod_{\substack{j \in \text{dom}(b_{d,t_d}) \\ (f_d \circ b_{d,t_d})(j) \in T^+}} (g_j \circ f_d \circ b_{d,t_d})(j) \in AA^{-1}.$$

Theorem 7.2 is the particular instance of 7.3 where T is the rooted tree with only two nodes.

Before proving Theorem 7.3 we recall some notation about ultrafilter limits. Let \mathcal{U} be an ultrafilter on a set S . Fix a topological space X , a function $f : S \rightarrow X$, and $x \in X$. Then we say that x is the \mathcal{U} -limit of f , in formulas $\mathcal{U} - \lim_s f(s) = x$, if for every neighborhood W of x in X , $\mathcal{U}s, f(s) \in W$. It follows from the properties of ultrafilters that, if X is a compact Hausdorff space, then any function $f : A \rightarrow X$ admits a unique \mathcal{U} -limit, which we also denote by $f(\mathcal{U})$.

Suppose that H is a Hilbert space. Then the set $\text{Ball}(B(H))$ of bounded linear operators on H of norm at most 1 is a compact semigroup with respect to the composition operation and the weak operator topology [8, Section I.3.1]. The group $U(H)$ of unitary operators on H is a subsemigroup of $\text{Ball}(B(H))$. Suppose now that S is a partial semigroup, and $v : S \rightarrow \text{Ball}(B(H))$, $s \mapsto v_s$ is a partial semigroup homomorphism. If $\mathcal{U} \in \gamma S$ is idempotent, then $v_{\mathcal{U}} = \mathcal{U} - \lim_s v_s$ is an orthogonal projection. Indeed, since \mathcal{U} is idempotent and

$v : \gamma S \rightarrow \text{Ball}(B(H))$ is a semigroup homomorphism, $v_{\mathcal{U}}$ is idempotent. Since $\|v_{\mathcal{U}}\| \leq 1$, $v_{\mathcal{U}}$ must be an orthogonal projection. As in the proof of [6, Corollary 2.1] one can deduce from this observation the following.

Lemma 7.4. Let (X, μ) be a probability space, and \tilde{A} be a measurable subset of X . Let $\text{Aut}(X, \mu)$ be the group of invertible measure-preserving transformations of (X, μ) . Suppose that S is an adequate partial semigroup, $S \rightarrow \text{Aut}(X, \mu)$, $s \mapsto \alpha_s$ is a partial semigroup homomorphism, and $\mathcal{U} \in \gamma S$ is idempotent. Then

$$\mathcal{U} - \lim_s \mu(\tilde{A} \cap \alpha_s(\tilde{A})) \geq \mu(\tilde{A})^2.$$

Proof. Let $H := L^2(X, \mu)$. For every $s \in S$ define $v_s \in U(H)$ by $v_s f = f \circ \alpha_s^{-1}$ for $f \in H$. This defines a partial semigroup homomorphism $S \rightarrow \text{Ball}(B(H))$, $s \mapsto v_s$. As observed above, $v_{\mathcal{U}}$ is an orthogonal projection. Let ξ be the vector of H corresponding to the characteristic function of \tilde{A} , and ξ_0 be the vector of H corresponding to the function constantly equal to 1. Observe that $v_s \xi_0 = \xi_0$ for every $s \in S$. Therefore $\langle v_{\mathcal{U}} \xi_0, \xi_0 \rangle = \|\xi_0\|^2$ and hence $v_{\mathcal{U}} \xi_0 = \xi_0$. Then we have that

$$\mathcal{U} - \lim_s \mu(\tilde{A} \cap \alpha_s(\tilde{A})) = \mathcal{U} - \lim_s \langle \xi, v_s \xi \rangle = \langle \xi, v_{\mathcal{U}} \xi \rangle = \|v_{\mathcal{U}} \xi\|^2 \geq \langle v_{\mathcal{U}} \xi, \xi_0 \rangle^2 = \langle \xi, \xi_0 \rangle^2 = \mu(A)^2.$$

□

We will now prove Theorem 7.3 using Lemma 7.4.

Proof of Theorem 7.3. Recall that the collection of adequate partial semigroup homomorphisms $\tau_f : \text{FIN}^T \rightarrow \text{FIN}^T$ when f ranges among the regressive homomorphisms of T , and the collection of adequate partial sub-semigroups FIN_t^T for $t \in T$, define a Ramsey action of T on FIN^T ; see Definition 3.3. Therefore by Theorem 3.4 and Proposition 2.5 there exists a function $\xi : T \rightarrow \gamma \text{FIN}^T$ such that $\xi(t) \in \gamma \text{FIN}_t^T$ is idempotent, $\xi(t) \leq \xi(t_0)$ for every $t, t_0 \in T$ with $t \leq t_0$, and $\tau_f \circ \xi = \xi \circ f$ for any regressive homomorphism $f : T \rightarrow T$.

Let (X, μ) and $\tilde{A} \subset X$ be, respectively, the standard probability space and the Borel set obtained from A applying Lemma 7.1. For $b \in \text{FIN}^T$ define

$$g_b := \prod_{i \in \text{dom}(b)} g_i(b(i)) \in G.$$

This gives a partial semigroup homomorphism $\text{FIN}^T \rightarrow G$, $b \mapsto g_b$. In turns, this gives a partial semigroup homomorphism $\text{FIN}^T \rightarrow \text{Aut}(X, \mu)$, $b \mapsto \tilde{g}_b$. Therefore by Lemma 7.4 we have that, for every $t \in T$,

$$d(A)^2 = \mu(\tilde{A})^2 \leq \xi(t) - \lim_{b \in \text{FIN}_t^T} \mu(\tilde{A} \cap \tilde{g}_b \tilde{A}) \leq \xi(t) - \lim_{b \in \text{FIN}_t^T} d(A \cap g_b A).$$

Observe that, whenever $d(A \cap gA) > 0$ one has that $A \cap gA \neq \emptyset$ and hence $g \in AA^{-1}$. Since $d(A) > 0$, we deduce in particular that, for every $t \in T$, $\xi(t)b, g_b \in AA^{-1}$. Observe now that the element

$$\prod_{d \in F} \prod_{\substack{j \in \text{dom}(b_{d,t_d}) \\ (f_d \circ b_{d,t_d})(j) \in T^+}} (g_j \circ f_d \circ b_{d,t_d})(j)$$

of G in the statement can be written, in the notation above, as

$$g_{\sum_{d \in F} \tau_{f_d}(b_{d,t_d})}.$$

Therefore the desired conclusion follows from Theorem 3.2. □

The same argument as the one in the proof of Theorem 7.3 gives the following generalization. Suppose as before that (g_n) is a sequence of functions $g_n : T^+ \rightarrow G$. Let also L be a subset of G . Adopting the notation from Subsection 4.3, we let FIN_L^T be the set of finitely-supported functions $b : \text{dom}(b) \rightarrow L \cup T^+$, where $\text{dom}(b)$ is a (possibly empty) finite subset of ω . For $t \in T^+$ we let $\text{FIN}_{L,t}^T$ be the set of $b \in \text{FIN}_L^T$ such that the intersection of the range of b with T is a nonempty chain with least element t . We also let $\text{FIN}_{L,r}^T$, where r is the root of T , be the set of $b \in \text{FIN}^T$ such that the range of b is contained in L . Recall the notion of variable substitution map $\sigma : \text{dom}(\sigma) \subset L \cup T^+ \rightarrow L \cup T^+$ with corresponding regressive homomorphism $f_\sigma : T \rightarrow T$ as in Definition 4.4. Any such map defines an adequate partial semigroup homomorphism $\tau_\sigma : \text{FIN}_L^T \rightarrow \text{FIN}_L^T$ obtained by setting, for $b \in \text{FIN}_L^T$, $\tau_\sigma(b) : \{n \in \text{dom}(b) : \sigma(b(n)) \in \text{dom}(\sigma)\} \rightarrow L \cup T^+$, $n \mapsto \sigma(b(n))$.

Theorem 7.5. Suppose that $A \subset G$ has positive density, and $L \subset G$. Let (g_n) be a sequence of functions $g_n : T^+ \rightarrow G$. Extend g_n to a function $g_n : T^+ \cup L \rightarrow G$ by setting $g_n(x) = x$. For any sequence (\mathcal{F}_n) of finite sets of variable substitution maps for L, T , there exists a sequence of functions $b_n : T \rightarrow \text{FIN}_L^T$, $t \mapsto b_{n,t} \in \text{FIN}_{L,t}^T$ such that $\text{dom}(b_{n,t}) < \text{dom}(b_{n+1,t})$ and for any finite nonempty subset F of ω , variable substitution maps $\sigma_d \in \mathcal{F}_d$ and nodes $t_d \in T$ for $d \in F$ such that $\{f_{\sigma_d}(t_d) : d \in F\}$ is a chain in T , one has that

$$\prod_{d \in F} \prod_{\substack{j \in \text{dom}(b_{d,t_d}) \\ b_{d,t_d}(j) \in \text{dom}(\sigma_d)}} (g_j \circ \sigma_d \circ b_{d,t_d})(j) \in AA^{-1}.$$

7.2. Polynomial configurations in delta sets in \mathbb{Z}^m . We now specialize the discussion in the case when G is the (additive) group \mathbb{Z}^m for some $m \in \mathbb{N}$. In this setting one can obtain a *polynomial* strengthening of Theorem 7.3. Denote by $\text{Int}[z_0, \dots, z_{m-1}]$ the space of polynomial functions $p : \mathbb{Z}^d \rightarrow \mathbb{Z}$ that are defined polynomials with rational coefficients in the variables z_0, \dots, z_{m-1} and which map integer points to integer points.

Theorem 7.6. Suppose $p = (p_0, \dots, p_{d-1}) \in \text{Int}[z_0, \dots, z_{m-1}]^d$. We will regard p as a function $p : \mathbb{Z}^m \rightarrow \mathbb{Z}^d$, and assume that $p(0) = 0$. Suppose that (g_n) is a sequence of functions $g_n : T^+ \rightarrow \mathbb{Z}^m$, and $A \subset \mathbb{Z}^d$ is a subset of positive density. There exists a sequence of functions $b_n : T \rightarrow \text{FIN}^T$, $t \mapsto b_{n,t} \in \text{FIN}_t^T$ such that $\text{dom}(b_{n,t}) < \text{dom}(b_{n+1,t})$ and for any finite nonempty subset F of ω , regressive homomorphisms $f_d : T \rightarrow T$ and nodes $t_d \in T$ for $d \in F$ such that $\{f_d(t_d) : d \in F\}$ is a chain in T , one has that

$$p \left(\sum_{d \in F} \sum_{\substack{j \in \text{dom}(b_{d,t_d}) \\ (f_d \circ b_{d,t_d})(j) \in T^+}} (g_j \circ f_d \circ b_{d,t_d})(j) \right) \in A - A.$$

Theorem 7.6 is a common generalization of Theorem 7.3 and [6, Corollary 2.2]. The proof of Theorem 7.6 is analogous to the proof of Theorem 7.3, where Lemma 7.4 is replaced by the following lemma, proved in [21, Theorem 6.1].

Lemma 7.7. Let (X, μ) be a probability space, and \tilde{A} be a measurable subset of X . Let T_0, \dots, T_{d-1} be pairwise commuting invertible measure-preserving transformations of (X, μ) . Suppose $p_0, \dots, p_{d-1} \in \text{Int}[z_0, \dots, z_{m-1}]$ are such that $p_i(0) = 0$ for $i \in d$. Suppose that $\mathcal{U} \in \beta\mathbb{Z}^m$ is an idempotent ultrafilter. Then

$$\mathcal{U} - \lim_{a \in \mathbb{Z}^m} \mu \left(\tilde{A} \cap \left(\prod_{i \in d} T_i^{p_i(a)} \right) (\tilde{A}) \right) \geq \mu(\tilde{A})^2.$$

We now present a proof of Theorem 7.6.

Proof of Theorem 7.6. Let the functions $\text{FIN}^T \rightarrow \mathbb{Z}^m$, $b \mapsto g_b$, and $\xi : T \rightarrow \gamma\text{FIN}^T$ be defined as in the proof of Theorem 7.3. Let also (X, μ) and \tilde{A} be obtained from A as in Lemma 7.1. The function $\text{FIN}^T \rightarrow \mathbb{Z}^m$, $b \mapsto g_b$, extends to a continuous semigroup homomorphism $\gamma\text{FIN}^T \rightarrow \beta\mathbb{Z}^m$, $\mathcal{V} \mapsto g_{\mathcal{V}}$. Set $\mathcal{U}_t := g_{\xi(t)}$ for $t \in T$, and observe that $\mathcal{U}_t \in \beta\mathbb{Z}^m$ is an idempotent ultrafilter. Let also e_0, \dots, e_{d-1} be the canonical generators of \mathbb{Z}^d , and $\tilde{e}_0, \dots, \tilde{e}_{d-1}$ be the corresponding elements of $\text{Aut}(X, \mu)$. Observe that

$$\prod_{i \in d} \tilde{e}_i^{p_i(a)} = \prod_{i \in d} \widetilde{p_i(a)} = \widetilde{p(a)}.$$

Therefore we have that, for every $t \in T$,

$$d(A)^2 \leq \mathcal{U}_t - \lim_{a \in \mathbb{Z}^m} \mu \left(\tilde{A} \cap \left(\prod_{i \in d} \tilde{e}_i^{p_i(a)} \right) \tilde{A} \right) = \lim_{b \in \text{FIN}^T} \mu \left(\tilde{A} \cap \widetilde{p(g_b)} \tilde{A} \right) \leq d(A \cap (p(g_b) + A)).$$

Therefore $\xi(t)b, p(g_b) \in A - A$. The conclusion follows again from Theorem 3.2. \square

In a similar fashion, one can prove the following more general result.

Theorem 7.8. Suppose $p = (p_0, \dots, p_{d-1}) \in \text{Int}[z_0, \dots, z_{m-1}]^d$ is such that $p(0) = 0$. Suppose that (g_n) is a sequence of functions $g_n : T^+ \rightarrow \mathbb{Z}^m$, and $A \subset \mathbb{Z}^d$ is a subset of positive density. Fix also a subset L of \mathbb{Z}^d . Let (\mathcal{F}_n) be a sequence of variable substitution maps for T, L . There exists a sequence of functions $b_n : T \rightarrow \text{FIN}_L^T$,

$t \mapsto b_{n,t} \in \text{FIN}_{L,t}^T$ such that $\text{dom}(b_{n,t}) < \text{dom}(b_{n+1,t})$ and for any finite nonempty subset F of ω , $\sigma_d \in \mathcal{F}_d$ and nodes $t_d \in T$ for $d \in F$ such that $\{f_{\sigma_d}(t_d) : d \in F\}$ is a chain in T , one has that

$$p \left(\sum_{d \in F} \sum_{\substack{j \in \text{dom}(b_{d,t_d}) \\ b_{d,t_d}(j) \in \text{dom}(\sigma_d)}} (g_j \circ \sigma_d \circ b_{d,t_d})(j) \right) \in A - A.$$

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